18.704 PROBLEM SET 8
TO BE DISCUSSED APR. 22, 27

Read Me

The following problems will all have the same format, where you answer the same questions below. The questions are independent of one another.

You will be given a semisimple Lie algebra $\mathfrak{g}$ that is naturally a subset of $N \times N$ matrices for some N. The Lie bracket is induced from the commutator on $\mathrm{Mat}_N(\mathbb{C})$. You will also be told what the Cartan subalgebra $\mathfrak{h}$ to use is. Then the questions are as follows:

(a) Find the decomposition of the adjoint representation $\mathfrak{g}$ into $\mathfrak{h}$-eigenspaces:
$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$
A.k.a. find the nonzero roots $\alpha \in \mathfrak{h}^*$ and description $\mathfrak{g}_\alpha$ in terms of matrices.

(b) Pick some $\ell \in \mathfrak{h}$ such that $\ell(\alpha) \neq 0$ for any root α. Let R be the set of roots. What is the set of *positive roots* $R^+ = \{\alpha \in R \mid \ell(\alpha) > 0\}$?

(c) Find the *simple roots* with respect to your choice of R^+ above: these are the α in R^+ that cannot be written as $\alpha_1 + \alpha_2$ for $\alpha_1, \alpha_2 \in R^+$.
(Hint: in each problem, there are only two simple roots.)

(d) There will be a natural embedding $\mathfrak{h} \hookrightarrow \mathbb{C}^N = \{$ diagonal matrices in $\mathrm{Mat}_N(\mathbb{C})\}$. Set $\mathfrak{h}_\mathbb{R} = \mathfrak{h} \cap \mathbb{R}^N$. The Euclidean inner product on $\mathbb{R}^N$ will then induce an inner product on $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}^*_\mathbb{R}$.
All the roots will lie in $\mathfrak{h}^*_\mathbb{R}$. Find the norms and angle between the two simple roots you found in (c).

(e) Draw a picture of all of the roots in a 2-dimensional plane, using (d).

(f) From the picture in (e), draw all the hyperplanes (a.k.a. lines) perpendicular to each root.

(g) CHALLENGE: what is the Weyl group a.k.a. the group generated by the reflections across all the hyperplanes from part (f)?

Problems

The *type* in parenthesis refers to the type of the associated Dynkin Diagram. For reference, $\mathfrak{sl}_3$ is type A_2, which is why you don't see it below.

1. (Type B_2)
Let $\mathfrak{g} = \mathfrak{so}(B) = \{X \in \mathrm{Mat}_5(\mathbb{C}) \mid X + B^{-1}X^TB = 0\}$ where X^T means transpose, and
$$B = \begin{pmatrix} 0 & \mathsf{Id}_2 & 0 \\ \mathsf{Id}_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
Equivalently, $\mathfrak{g} = \{X \mid (BX) = -(BX)^T\}$. [1]

[1] This says that $\mathfrak{so}(B)$ is isomorphic to $\mathfrak{so}_5 := \mathfrak{so}(\mathsf{Id}_5)$.

The Cartan subalgebra is

$$\mathfrak{h} = \mathfrak{g} \cap \{\text{diagonal matrices}\} = \{\text{diag}(x_1, x_2, -x_1, -x_2, 0)\}$$

which is 2-dimensional. Note this only works in the definition of $\mathfrak{so}(B)$ with B as above – the intersection is zero if $B = \mathsf{Id}_5$.

2. (Type C_2)
Let $\mathfrak{g} = \mathfrak{sp}_2(\mathbb{C}) = \{X \in \text{Mat}_4(\mathbb{C}) \mid X + J^{-1}X^TJ = 0\}$ where X^T means transpose, and

$$J = \begin{pmatrix} 0 & \mathsf{Id}_2 \\ -\mathsf{Id}_2 & 0 \end{pmatrix}$$

The Cartan subalgebra is

$$\mathfrak{h} = \mathfrak{g} \cap \{\text{diagonal matrices}\} = \{\text{diag}(x_1, x_2, -x_1, -x_2)\}$$

which is 2-dimensional.

3. (Type D_2)
Let $\mathfrak{g} = \mathfrak{so}(B) = \{X \in \text{Mat}_4(\mathbb{C}) \mid X + B^{-1}X^TB = 0\}$ where X^T means transpose, and

$$B = \begin{pmatrix} 0 & \mathsf{Id}_2 \\ \mathsf{Id}_2 & 0 \\ 0 & 0 \end{pmatrix}$$

Equivalently, $\mathfrak{g} = \{X \mid (BX) = -(BX)^T\}$. [2]
The Cartan subalgebra is

$$\mathfrak{h} = \mathfrak{g} \cap \{\text{diagonal matrices}\} = \{\text{diag}(x_1, x_2, -x_1, -x_2)\}$$

which is 2-dimensional. Note this only works in the definition of $\mathfrak{so}(B)$ with B as above – the intersection is zero if $B = \mathsf{Id}_4$.

[2]This says that $\mathfrak{so}(B)$ is isomorphic to $\mathfrak{so}_4 := \mathfrak{so}(\mathsf{Id}_4)$.