# 18.704 PROBLEM SET 7
## TO BE DISCUSSED APR. 8, 15

Let $\mathfrak{g} = \mathfrak{sl}_3$ throughout this problem set.

**1.** Consider the $\mathfrak{g}$-representation $V = \mathbb{C}^3$ where $\mathfrak{g} = \mathfrak{sl}_3$ acts by matrix multiplication. Define a linear map $\Phi : V \otimes V^* \to \mathbb{C}$ by $v \otimes \xi \mapsto \xi(v)$.

(a) Show that $\Phi$ is a map of $\mathfrak{sl}_3-$representations, where $\mathbb{C}$ is given the trivial representation ($xz := 0$ for all $x \in \mathfrak{sl}_3$, $z \in \mathbb{C}$).

(b) Recall that there is a natural identification $\alpha : V \otimes V^* \cong \mathrm{Hom}(V, V)$, given by $v \otimes \xi \mapsto (w \mapsto \xi(w)v)$. Show that under this identification, $\Phi \circ \alpha^{-1}$ sends a linear operator $T \in \mathrm{Hom}(V, V)$ to its trace $\mathrm{tr}(T)$.

(c) Prove that $\ker \Phi$ is isomorphic (as $\mathfrak{sl}_3-$representations) to the adjoint representation of $\mathfrak{sl}_3$.

**2.** With $V = \mathbb{C}^3$ as above, define a linear map $\Psi : \mathrm{Sym}^2 V \otimes V^* \to V$ by

$$(u \otimes v) \otimes \xi \mapsto u\,\xi(v) + v\,\xi(u).$$

Show that $\Psi$ is a map of $\mathfrak{sl}_3-$representations.

**3.** Let $V = \mathbb{C}^3$ as above, and $V^*$ its dual representation. Let $e_1, e_2, e_3$ be the standard basis of $V$, and $e_1^*, e_2^*, e_3^*$ the standard dual basis, so $e_i^*(e_j) = \delta_{ij}$.

Let $a \geq 0, b \geq 0$ be integers.

(a) Show that $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ has a basis given by

$$e^I \otimes (e^*)^J := e_1^{i_1} e_2^{i_2} e_3^{i_3} \otimes (e_1^*)^{j_1} (e_2^*)^{j_2} (e_3^*)^{j_3}$$

where $i_1, i_2, i_3, j_1, j_2, j_3 \geq 0$ are all integers and $i_1 + i_2 + i_3 = a$, $j_1 + j_2 + j_3 = b$. We abbreviate $I = (i_1, i_2, i_3)$ and $J = (j_1, j_2, j_3)$ for such triples. Denote $\deg I = i_1 + i_2 + i_3$ and $\deg J = j_1 + j_2 + j_3$ (so in the above $\deg I = a, \deg J = b$.

(b) Show the weight of $e^I \otimes (e^*)^J$ is

$$(i_1 - j_1 - i_2 + j_2)L_1 + (i_2 - j_2 - i_3 + j_3)(-L_3) \in \mathfrak{h}^*.$$

(c) From parts (a)-(b) deduce that the dimension (a.k.a. “multiplicity”) of each nonzero weight space of $\mathrm{Sym}^a V$ is equal to 1.

(d) Draw the weights of the representation $\mathrm{Sym}^5 V \otimes \mathrm{Sym}^3 V^*$ and then using part (b) find the multiplicities of the weight spaces.

**4.** All notation is the same as in Problem 3. The goal is to show that there is exactly one dimension of highest weight vector of weight $(a - i)L_1 + (b - i)(-L_3)$ in $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$, for $i \leq \min(a, b)$. (This is proof of Claim 13.4 in Fulton–Harris.)

(a) Show, either using 3(b) or geometrically, that if $e^I \otimes (e^*)^J$ has weight $(a-i)L_1 + (b - i)(-L_3)$, then

$$e^I \otimes e^J = (e_1^{a-i} e^{I'}) \otimes ((e_3^*)^{b-i} (e^*)^{I'})$$

for a new triple $I'$ with $\deg(I') = i$.

(b) Now suppose $v$ is a highest weight vector of weight $(a-i)L_1 + (b-i)(-L_3)$ of $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$. Then by (a), it must be a linear combination

$$v = \sum_{\deg(I)=i} c_I \cdot (e_1^{a-i} e^I) \otimes ((e_3^*)^{b-i} (e^*)^I)$$

(here $I$ replaced what was formerly $I'$).

Highest weight vector means that $E_{12}v = 0$ and $E_{23}v = 0$. By explicit computation of $E_{12}v$ and $E_{23}v$ in terms of the definition of the representation, conclude that we must have all $c_I$ are equal.

*Conclusion:* Thus $v$ is unique up to scalar multiplication.

### Weyl character formula

Now we want to find the multiplicities of the irreducible representations $\Gamma_{a,b}$. A convenient way to keep track of the multiplicities of a finite dimensional representation $V$ of $\mathfrak{g}$ is via the *character*[1] $\mathrm{ch}(V)$ defined as a formal sum

$$\mathrm{ch}(V) = \sum_{\lambda \in \Lambda_W} (\dim V[\lambda]) t^\lambda$$

where the sum is over all elements of the weight lattice, and $t^\lambda$ is just a formal symbol for a basis element in the big infinite dimensional vector space $\mathbb{C}^{\Lambda_W}$. We understand $0 \cdot e^\lambda = 0$, so the sum above is really a sum over the nonzero weights of $V$, and we attach a number (the dimension of the weight space $V[\lambda]$) to each $\lambda$. This definition of character is similar to that of a *generating function*, if you have seen that term before.

For further reading, see [Kirillov, Definition 8.4], [Fulton–Harris, §23.2].

In the case of $\mathfrak{sl}_3$, we will be slightly more concrete: any element $\lambda \in \Lambda_W$ can be written as $a_1L_1 + a_2L_2 + a_3L_3$ for $a_i \in \mathbb{Z}$, where $L_1 + L_2 + L_3 = 0$. We let $x_i := t^{L_i}$, and we think of $t^\lambda$ as a monomial, so

$$t^{a_1L_1+a_2L_2+a_3L_3} = x_1^{a_1} x_2^{b_1} x_3^{c_1}$$

together with the relation $x_1x_2x_3 = 1$. In other words, the characters of finite dimensional representations live in the polynomial algebra

$$\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]/(x_1x_2x_3 - 1).$$

(The $x_3$ above is extraneous, since $x_3 = x_1^{-1}x_2^{-1}$, but it is convenient to keep it in the notation so the formulas look nicer.)

**5.** Let $a, b \geq 0$ be integers. Prove, using the fact that

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^* = \bigoplus_{i=0}^{\min(a,b)} \Gamma_{a-i,b-i},$$

that the character of the irreducible representation $\Gamma_{a,b}$ is

$$\mathrm{ch}(\Gamma_{a,b}) = \frac{\sum_{\sigma \in S_3} \mathrm{sign}(\sigma) x_{\sigma(1)}^{a+2} x_{\sigma(2)}^{b+1} x_{\sigma(3)}^{0}}{\sum_{\sigma \in S_3} \mathrm{sign}(\sigma) x_{\sigma(1)}^{2} x_{\sigma(2)}^{1} x_{\sigma(3)}^{0}} \tag{1}$$

[1]The term *character* refers to various related but slightly different things in representation theory. This is just to remark that if you see "character" somewhere else later, it might not mean exactly what is said here.

In the above, $S_3$ is the symmetric group, a.k.a. the set of permutations on $\{1, 2, 3\}$. So we think of a permutation $\sigma$ as a bijection $\{1, 2, 3\} \to \{1, 2, 3\}$ and $\sigma(i)$ means where this bijection sends $i$.

Also, $x^0_{\sigma(3)} = 1$ above: I included it only for clarity of the pattern. The formula (1) can also be written using determinants of $3 \times 3$ matrices (recall one formula for determinant involves a sum over permutations):

$$\mathrm{ch}(\Gamma_{a,b}) = \frac{\det\begin{pmatrix} x_1^{a+2} & x_1^{b+1} & 1 \\ x_2^{a+2} & x_2^{b+1} & 1 \\ x_3^{a+2} & x_2^{b+1} & 1 \end{pmatrix}}{\det\begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{pmatrix}}$$

**Note:** that the denominator of the previous formula is the *Vandermonde matrix*, which is known to have determinant equal to $\prod_{1\le i<j\le 3}(x_i - x_j)$. This is a special case of the *Weyl denominator identity.*

**Remark.** What is **amazing** above is that although the right hand side of (1) is a *fraction*, since the left hand side is the character of a representation, the right hand side must be a *polynomial.* The polynomial $\mathrm{ch}(\Gamma_{a,b})$ is called a **Schur polynomial**. It shows up in many other areas of math.

**General Weyl Character Formula:** There is a natural generalization of the above formula for representations of $\mathfrak{sl}_n$. To generalize it further to an arbitrary semisimple Lie algebra:

The significance of $\rho = (2, 1, 0)$ above is that it is 1/2 of the sum of the positive roots: $\rho = \frac{1}{2}(L_1 - L_2 + L_2 - L_3 + L_1 - L_3) \in \mathfrak{h}^*$. The symmetric group $S_3$ is the *Weyl group* of $\mathfrak{sl}_3$.

Now for an arbitrary semisimple Lie algebra $\mathfrak{g}$, its irreducible representations are indexed by the $\lambda \in \Lambda_W$ that lie in the dominant Weyl chamber (after making a choice of positive roots). The Weyl character formula says

$$\mathrm{ch}(\Gamma_\lambda) = \frac{\sum_{w\in W} \mathrm{sign}(w) t^{w(\lambda+\rho)}}{\prod_{\alpha\in R^+}(t^{\alpha/2} - t^{-\alpha/2})}$$

where $W$ is the Weyl group, $R^+$ is the set of positive roots, and $\mathrm{sign}(w)$ is the sign of an element in $W$ (which we did not define).