18.704 PROBLEM SET 6
DUE: APR. 3, 11:59PM EDT

1. This problem is about justifying why the pictures of lattices you see in the book look correct. It has nothing to do with Lie algebras; it is more like a geometry problem from 18.02.

Let $\mathfrak{h}_{\mathbb{R}} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid a_1 + a_2 + a_3 = 0\} \subset \mathbb{R}^3$. Let $\mathfrak{h}_{\mathbb{R}}^*$ denote the $\mathbb{R}$-vector space of $\mathbb{R}$-linear functionals $L : \mathfrak{h}_{\mathbb{R}} \to \mathbb{R}$.

We have an isomorphism $\phi : \mathbb{R}^3 \to (\mathbb{R}^3)^*$ by sending

$$\phi(v) = (w \mapsto \langle v, w \rangle) \in (\mathbb{R}^3)^*, \qquad v, w \in \mathbb{R}^3$$

where $\langle v, w \rangle$ denotes the dot product.

(a) We get a map $\phi' : \mathfrak{h}_{\mathbb{R}} \to (\mathfrak{h}_{\mathbb{R}})^*$ by sending $H \in \mathfrak{h}_{\mathbb{R}}$ to the restriction of the functional $\phi(H) : \mathbb{R}^3 \to \mathbb{R}$ to $\phi(H)|_{\mathfrak{h}_{\mathbb{R}}} : \mathfrak{h}_{\mathbb{R}} \to \mathbb{R}$. Show that ϕ' is an isomorphism of $\mathbb{R}$-vector spaces.

(b) We have $L_1, L_2 \in \mathfrak{h}_{\mathbb{R}}^*$ defined by $L_i(a_1, a_2, a_3) = a_i$ for $i = 1, 2$. What is the angle between the vectors $\phi'^{-1}(L_1), \phi'^{-1}(L_2) \in \mathfrak{h}_{\mathbb{R}} \subset \mathbb{R}^3$, in terms of usual Euclidean geometry on $\mathbb{R}^3$? What are the lengths of the two vectors $\phi'^{-1}(L_1), \phi'^{-1}(L_2)$?

[Compare your answers to the pictures in Fulton–Harris.]

2. Let $1 \leq i < j \leq 3$. Define the $\mathbb{C}$-vector subspace $S_{ij} \subset \mathfrak{sl}_3$ as the span of E_{ij}, H_{ij}, E_{ji}, where $H_{ij} = [E_{ij}, E_{ji}]$. Show that S_{ij} is a Lie subalgebra isomorphic to $\mathfrak{sl}_2$.

3. Let V be a finite-dimensional representation of $\mathfrak{sl}_3$. Let $1 \leq i < j \leq 3$ and consider the Lie subalgebra $S_{ij} \subset \mathfrak{sl}_3$ from Problem 2. We consider V as a representation of S_{ij} by restriction.

For a fixed $\alpha \in \Lambda_W \subset \mathfrak{h}^*$, define the vector subspace

$$W = \bigoplus_{k \in \mathbb{Z}} V[\alpha + k(L_j - L_i)]$$

where $V[\lambda]$ is the weight space of all $\mathfrak{h}$-eigenvectors with eigenvalue λ ($V[\lambda]$ is denoted V_λ in Fulton–Harris).

(a) Show that W is an S_{ij}-subrepresentation of V.

Deduce from $\mathfrak{sl}_2$ representation theory that $\alpha(H_{ij}) \in \mathbb{Z}$ and the set of

$$\{k \in \mathbb{Z} \mid V[\alpha + k(L_j - L_i)] \neq 0\}$$

must equal $\{k \in \mathbb{Z} \mid -N \leq \alpha(H_{ij}) - 2k \leq N\}$ for some integer $N \geq 0$.

(Note I really mean every integer in the interval above; you should ask yourself where did the $2\mathbb{Z}$ in $\mathfrak{sl}_2$ representation theory go?)

(b) Consider the line $\{\alpha + x(L_j - L_i) \mid x \in \mathbb{R}\} \subset \mathfrak{h}_{\mathbb{R}}^*$. Show that this line is orthogonal to the line $\{L \in \mathfrak{h}_{\mathbb{R}}^* \mid L(H_{ij}) = 0\}$.

(Hint: what do two lines being orthogonal in $\mathfrak{h}_{\mathbb{R}}^*$ mean? Answer: it means you consider $\mathfrak{h}_{\mathbb{R}}^*$ as a subspace of $\mathbb{R}^3$ using ϕ'^{-1} from Problem 1, and then use the usual Euclidean notion of orthogonality.)

(c) Show that even if V is irreducible as a $\mathfrak{sl}_3$-representation, it is NOT always true that W is irreducible, by doing the following example:

Let $V = \mathfrak{sl}_3$ the adjoint representation. Let $\alpha = 0$. Let $i = 1, j = 2$. Find the decomposition of W as a $S_{12} \cong \mathfrak{sl}_2$ representation into irreducible representations.

4. Recall that the root lattice $\Lambda_R \subset \mathfrak{h}^*$ of $\mathfrak{sl}_3(\mathbb{C})$ is the lattice generated by $L_1 - L_2, L_1 - L_3, L_1 - L_3$ and that the weight lattice $\Lambda_W \subset \mathfrak{h}^*$ of $\mathfrak{sl}_3(\mathbb{C})$ is the lattice generated by L_1, L_2, L_3.

(a) Show that

$$\Lambda_W = \{L \in \mathfrak{h}^* \mid L(H_{ij}) \in \mathbb{Z} \text{ for all } 1 \leq i < j \leq 3\}$$

where $H_{ij} = [E_{ij}, E_{ji}] \in \mathfrak{h}$. In coordinates, $H_{12} = (1, -1, 0)$ for example.

(Hint: pick a basis and write down the conditions.)

(b) Let W be the $S_{ij} = \mathfrak{sl}_2$ representation from Problem 3. Show that $V[\alpha] \subset W$ is an eigenspace for H_{ij} with eigenvalue equal to $\alpha(H_{ij}) \in \mathbb{C}$.

Conclusion[1]: However from $\mathfrak{sl}_2$ representation theory we know that all eigenvalues are integers, so $\alpha(H_{ij}) \in \mathbb{Z}$. This is true for any weight α of V and any pair i, j, so we have shown any weight α of V satisfies $\alpha(H_{ij}) \in \mathbb{Z}$ for all i, j. By part (a), this implies that $\alpha \in \Lambda_W$.

[1]this is for you to read, not to prove