18.704 PROBLEM SET 5
DUE: MAR. 26, 11:59PM EDT

1. Let V be a finite-dimensional $\mathbb{C}$-vector space. Suppose that $A, B \in \mathrm{End}(V)$ commute (i.e., $AB = BA$), and each of A, B is diagonalizable (i.e., there exists a basis of V consisting of eigenvectors). Show that A and B are simultaneously diagonalizable, i.e., show that there exists a basis of V consisting of vectors each of which is both an eigenvector for A and B.

2. Let $\mathfrak{g}_\alpha$ be an eigenspace of $\mathfrak{h}$ in $\mathfrak{sl}_3$ with $\alpha \neq 0$. Are there any eigenspaces for $\mathfrak{g}_\alpha$?

3. Recall that for any finite-dimensional $\mathfrak{sl}_3$-representation V, the eigenvalues differ by integer linear combinations of $\alpha_{i,j} := L_i - L_j \in \mathfrak{h}^*$ for $1 \leq i, j \leq 3$.

Let $\lambda \in \mathfrak{h}^*$ be the *maximal* eigenvalue, where we say $\mu < \lambda$ if $\lambda - \mu \in \sum_{i<j} \mathbb{Z}_{\geq 0}\alpha_{i,j}$. Find λ when

(a) $V = \mathbb{C}^3$, where the action $\rho : \mathfrak{sl}_3 \to \mathrm{End}(\mathbb{C}^3)$ is the natural inclusion.
(b) $V = (\mathbb{C}^3)^*$, where $\mathbb{C}^3$ is as in (a) and $(\mathbb{C}^3)^*$ is its dual representation.
(c) V is the adjoint representation of $\mathfrak{sl}_3$.

4. Fix real numbers $a_1, a_2, a_3 \in \mathbb{R}$ such that $a_1 + a_2 + a_3 = 0$ and $a_1 > a_2 > a_3$. (A typical example is $a_1 = 1, a_2 = 0, a_3 = -1$.)

Prove that the linear functional $\ell : \mathfrak{h}^* \to \mathbb{C}$ defined by

$$\ell(b_1L_1 + b_2L_2 + b_3L_3) = a_1b_1 + a_2b_2 + a_3b_3$$

has the following properties:

(a) The roots $\alpha = L_i - L_j$ for which $\ell(\alpha) > 0$ are exactly $\{L_i - L_j \mid 1 \leq i < j \leq 3\}$.
(b) ℓ has real values on the root lattice $\Lambda_R \subset \mathfrak{h}^*$.
(Remark: the roots $\alpha_{1,2}, \alpha_{2,3}$ form a basis of $\mathfrak{h}$, so defining ℓ on $\Lambda_R \to \mathbb{R}$ is equivalent to defining ℓ on $\mathfrak{h}^* \to \mathbb{C}$.)

5. Let V be a finite dimensional representation of $\mathfrak{sl}_3$. Prove that there exists a *highest weight vector* in V.

6. Let v be a highest weight vector V of an irreducible representation.

We will define W_i as the set of elements generated with **at most** i successive applications of $E_{2,1}, E_{3,2} \in \mathfrak{sl}_3$ (note we do NOT include $E_{3,1}$) to the highest weight vector v. Note that $\bigcup_{i\geq 0} W_i = V$ because $E_{3,1} = [E_{3,2}, E_{2,1}]$.

Prove by induction that the action of the elements $E_{1,2}, E_{2,3} \in \mathfrak{sl}_3$ sends W_i to W_{i-1}.

Again because $[E_{1,2}, E_{2,3}] = E_{1,3}$, the above implies that $\bigcup_i W_i = V$ is stable under the action of $\mathfrak{sl}_3$. For reference, this is the proof of [Fulton–Harris, Claim 12.10].

7. Let $\lambda \in \mathfrak{h}^*$ be the eigenvalue of the highest weight vector in an irreducible finite-dimensional $\mathfrak{sl}_3$ representation V. Prove that $V[\lambda]$ is one-dimensional. You can assume [Fulton–Harris, Claim 12.10].