18.704 PROBLEM SET 4
DUE: MAR. 19, 11:59PM EDT

For this entire problem set, $\mathfrak{g} = \mathfrak{sl}_2$.

Definition. For $\lambda \in \mathbb{C}$, let M_λ denote the Verma module of $\mathfrak{sl}_2$ of highest weight λ. Let L_λ denote the irreducible quotient representation of M_λ.

From HW3, you know that $L_\lambda = M_\lambda$ if $\lambda \notin \mathbb{Z}_{\geq 0}$ and $L_\lambda = V_\lambda$ when $\lambda \in \mathbb{Z}_{\geq 0}$.

1. Show that Verma modules have the following property: for any $\lambda \in \mathbb{C}$ and $U(\mathfrak{g})$-module V, we have

$$\mathrm{Hom}_{\mathfrak{g}}(M_\lambda, V) = \mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda, V) = \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{b}} V).$$

Here $\mathfrak{b} \subset \mathfrak{g}$ is the Lie subalgebra spanned by h, e and for any $\mathfrak{g}$-representation V, we can simply considered it as a $\mathfrak{b}$-representation $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{b}} V$ by restricting the action of $\mathfrak{g}$ to the subalgebra $\mathfrak{b}$.

(Hint: use HW3, Problem 3,5.)

Definition. A $U(\mathfrak{g})$-module M is a **highest weight representation** (or module) of highest weight λ if there is a vector $v^+ \in M[\lambda]$ such that M is spanned by $f^i v^+$ for $i = 0, 1, \ldots, \infty$.

The representations M_λ and L_λ are all highest weight representations.

2. Using Problem 1, deduce that:

(a) Any highest weight representation of highest weight λ is a quotient of M_λ.

(b) Any irreducible, highest weight representation is isomorphic to L_λ for some $\lambda \in \mathbb{C}$.

3. Recall the Casimir element $C = \frac{1}{2}H^2 + FE + EF \in U(\mathfrak{g})$. Show that C acts on M_λ as scalar multiplication $\chi_\lambda \cdot \mathrm{Id}$ where $\chi_\lambda = \frac{\lambda(\lambda+2)}{2}$. (Note that it is NOT automatic that C acts by a scalar when M_λ is not irreducible.)

4. Let $\lambda, \mu \in \mathbb{C}$ such that $\mu - \lambda$ is NOT in $\mathbb{R}_{>0}$ (i.e., $\lambda \nless \mu$). Suppose you have a short exact sequence

$$0 \to M_\mu \to V \to M_\lambda \to 0$$

of $\mathfrak{g}$-representations, such that the action of $H \in \mathfrak{g}$ on V is diagonalizable (i.e., V decomposes into H-eigenspaces). Show that V must be isomorphic to $M_\mu \oplus M_\lambda$ as representations.