18.704 PROBLEM SET 3
DUE: MAR. 12, 11:59PM EST

Tensor Products

1. Let V be a *finite dimensional* $\mathbb{C}$ vector space. Let W be a possibly infinite dimensional $\mathbb{C}$ vector space. We have a bilinear map

$$V^* \times W \to \operatorname{Hom}_{\mathbb{C}}(V, W)$$

sending $(\xi, w) \in V^* \times W$ to the linear operator $T_{\xi,w} : V \to W$ defined by $T_{\xi,w}(v) = \xi(v)w$ for $v \in V$. By the universal property for tensor product, this $\mathbb{C}$-bilinear map extends uniquely to a $\mathbb{C}$-linear map $\Phi : V^* \otimes W \to \operatorname{Hom}_{\mathbb{C}}(V, W)$.

Show that Φ is an isomorphism.

(Not to be submitted: ask yourself what goes wrong with your proof if V is infinite dimensional. If V is infinite dimensional, Φ is NOT an isomorphism.)

2. Let $\mathfrak{g}$ be a Lie algebra and (V, ρ_V) a $\mathfrak{g}$ representation.

(a) Define $\rho_{V^*} : \mathfrak{g} \to \operatorname{End}(V^*)$ by

$$(\rho_{V^*}(x)\xi)(v) := -\xi(\rho_V(x)v), \qquad x \in \mathfrak{g}, \xi \in V^*, v \in V$$

where $\rho_{V^*}(x)\xi \in V^*$. Show that (V^*, ρ_{V^*}) is a representation.

(b) Let (W, ρ_W) be another $\mathfrak{g}$ representation. Define $\rho_{V \otimes W} : \mathfrak{g} \to \operatorname{End}(V \otimes W)$ by

$$\rho_{V \otimes W}(x)(v \otimes w) := (\rho_V(x)v) \otimes w + v \otimes (\rho_W(x)w), \qquad x \in \mathfrak{g}, v \in V, w \in W.$$

(This defines $\rho_{V \otimes W}(x)$ on pure tensors $v \otimes w$, and the definition extends linearly to a linear operator $V \otimes W \to V \otimes W$ by the universal property.)

Show that $(V \otimes W, \rho_{V \otimes W})$ is a representation.

(c) Let V, W be representations. Assume that V is finite dimensional. By parts (a)-(b), we have a representation $V^* \otimes W$. Let

$$(V^* \otimes W)^{\mathfrak{g}} := \{u \in V^* \otimes W \mid \rho(x)u = 0 \text{ for all } x \in \mathfrak{g}\}.$$

Note that u does NOT need to be a pure tensor above.

From Problem 1 we have an isomorphism

$$\Phi : V^* \otimes W \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{C}}(V, W).$$

Show that the image of $(V^* \otimes W)^{\mathfrak{g}} \subset V^* \otimes W$ under Φ is equal to $\operatorname{Hom}_{\mathfrak{g}}(V, W) \subset \operatorname{Hom}_{\mathbb{C}}(V, W)$, the subset of maps of representations $V \to W$.

3. (Etingof, Problem 2.11.6)

Let A, B be two unital associative $\mathbb{C}$-algebras. An (A, B)**-bimodule** is defined to be a $\mathbb{C}$-vector space V with both a left A-module structure and a right B-module structure which satisfy $(av)b = a(vb)$ for any $v \in V, a \in A, b \in B$ (i.e., the left action of A commutes with the right action of B).

Let V be an (A, B)-bimodule and let W be a left B-module. Then $V \otimes_B W$ has the structure of a left A-module by $a(v \otimes w) := (av) \otimes w$ for $a \in A, v \in V, w \in W$.

Let U be a left A-module. Let $\operatorname{Hom}_A(V, U)$ denote the vector space of all maps $V \to U$ of left A-modules. Then $\operatorname{Hom}_A(V, U)$ naturally has a structure of **left** B-module, defined by setting $(b\phi)(v) := \phi(vb)$ for all $b \in B, \phi \in \operatorname{Hom}_A(V, U), v \in V$ using the **right** B-module structure on V.

Using the universal property of tensor product (of modules), prove that there is a natural isomorphism

$$\Psi : \mathrm{Hom}_A(V \underset{B}{\otimes} W, U) \xrightarrow{\sim} \mathrm{Hom}_B(W, \mathrm{Hom}_A(V, U)).$$

Hint: the map Ψ is defined from left to right by sending

$$f \mapsto (w \mapsto (v \mapsto f(v \otimes w)))$$

for $f \in \mathrm{Hom}_A(V \otimes_B W, U), w \in W, v \in V$.

Verma Modules

4. (Poincaré–Birkhoff–Witt theorem for $\mathfrak{sl}_2$)

Show that the universal enveloping algebra $U(\mathfrak{sl}_2)$ has a basis as a $\mathbb{C}$-vector space given by all monomials $f^i h^j e^k$ ranging over all non-negative integers i, j, k.

(Recall that a basis of an infinite dimensional vector space means that any vector can be written *uniquely* as a *finite* linear combination of basis vectors.)

5. Recall from [Kirillov, Lemma 4.58] that for $\lambda \in \mathbb{C}$, the Verma module M_λ is defined to be the infinite-dimensional vector space with basis $v^0, v^1, v^2, \ldots$ such that the $\mathfrak{sl}_2$-action is defined by $ev^0 = 0$ and

$$\begin{aligned} hv^k &= (\lambda - 2k)v^k \\ fv^k &= (k+1)v^{k+1} \\ ev^k &= (\lambda - k + 1)v^{k-1}, \quad k > 0. \end{aligned}$$

Let $\mathfrak{b} = \mathrm{span}_{\mathbb{C}}(h, e)$. This is a Lie subalgebra of $\mathfrak{g}$. Let $\mathbb{C}_\lambda$ denote the one-dimensional $\mathfrak{b}$-representation where the underlying vector space is $\mathbb{C}$ and

$$hv = \lambda v,\ ev = 0 \quad \text{for } v \in \mathbb{C}.$$

Then $\mathbb{C}_\lambda$ is a left $U(\mathfrak{b})$-module. We can consider $U(\mathfrak{sl}_2)$ as a $(U(\mathfrak{sl}_2), U(\mathfrak{b}))$-biomodule where $U(\mathfrak{sl}_2)$ acts by left multiplication and $U(\mathfrak{b}) \subset U(\mathfrak{sl}_2)$ acts by right multiplication. From Problem 3, we saw that $U(\mathfrak{sl}_2) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ is a left $U(\mathfrak{sl}_2)$-module.

Show that there is a natural isomorphism of $U(\mathfrak{sl}_2)$-modules (equivalently, $\mathfrak{sl}_2$-representations)

$$U(\mathfrak{sl}_2) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda \xrightarrow{\sim} M_\lambda$$

where $1 \otimes 1 \mapsto v^0$.

Hint: Use Problem 4. The same argument as in Problem 4 also says $U(\mathfrak{b})$ has a basis by all monomials $h^j e^k$ ranging over all non-negative integers j, k.

6. Let $\lambda = n$ be a non-negative integer, and $M_\lambda = M_n$ denotes the Verma module. Define the infinite subspace $W \subset M_n$ to be the span of $v^{n+1}, v^{n+2}, \ldots$.

(a) Show that W is a $\mathfrak{sl}_2$-subrepresentation.
(b) Show that W is isomorphic to M_{-n-2} as representations.
(c) Show that the quotient representation M_n/W is isomorphic to the irreducible finite dimensional representation V_n defined previously.

7. Show that if $\lambda \in \mathbb{C}$ is not a non-negative integer, then M_λ is an irreducible representation.