18.704 PROBLEM SET 2
DUE: MAR. 5, 11:59PM EST

1. Fix a nonnegative integer n. Recall that we defined V_n to be a $\mathbb{C}$-vector space with basis $\{v_0, v_1, \dots, v_n\}$, and we defined a $\mathbb{C}$-linear map $\rho_n : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V_n)$ by

$$\rho_n(h)v_k := (n-2k)v_k,$$
$$\rho_n(f)v_k := (k+1)v_{k+1},$$
$$\rho_n(e)v_k := (n+1-k)v_{k-1}$$

where in the last two definitions, we have the edge cases $\rho_n(f)v_n := 0$ and $\rho_n(e)v_0 := 0$. Check that this actually defines a representation of $\mathfrak{sl}_2(\mathbb{C})$.

2. Let V be the finite dimensional $\mathfrak{sl}_2$ representation with dimensions of weight spaces $V[k]$ given by the following table:

k :	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\dim V[k] =$	1	0	3	3	3	3	3	3	3	0	1

and all weight spaces not shown have dimension 0.

Find the decomposition of V into a direct sum of irreducible representations $\bigoplus_i V_{n_i}$, where V_{n_i} is as in Problem 1.

3. Let I be a two-sided ideal in an associative algebra A.

(a) Let $\pi : A \to A/I$ be the quotient map. Prove that

$$\pi(a)\pi(b) := \pi(ab), \qquad a, b \in A$$

is a consistent definition of multiplication in A/I and conclude that A/I is an algebra.

(b) Let V be an A-module and $W \subset V$ an A-submodule. Let $\pi : V \to V/W$ be the quotient map. Prove that

$$a \cdot \pi(v) := \pi(a \cdot v), \qquad a \in A, v \in V$$

is a consistent definition of an A-module structure on V/W.

4. Define the Casimir element

$$C := ef + fe + \frac{1}{2}h^2 \in U(\mathfrak{sl}_2).$$

Show that C commutes with every other element in $U(\mathfrak{sl}_2)$.

5. Let A be a unital associative algebra over $\mathbb{C}$. Let $\phi : V_1 \to V_2$ be a map of A-modules. Prove that if V_2 is irreducible and ϕ is nonzero, then ϕ is surjective.

6. Let V_n be the irreducible representation of $\mathfrak{sl}_2$ from Problem 1. We can think of V_n as a $U(\mathfrak{sl}_2)$ module, which is the same as saying that the $\mathbb{C}$-linear map $\rho_n : \mathfrak{sl}_2 \to \mathrm{End}(V_n)$ extends to a homomorphism of associative algebras

$$\rho_n : U(\mathfrak{sl}_2) \to \mathrm{End}(V_n),$$

also denoted by ρ_n.

(a) Use Schur's lemma to show that $\rho_n(C) = \alpha_n \operatorname{Id}_{V_n}$, i.e., C acts on V_n by a scalar $\alpha_n \in \mathbb{C}$, where C denotes the Casimir element.

(b) Find the scalar $\alpha_n \in \mathbb{C}$.

7. (Complete reducibility of finite dimensional representations of $\mathfrak{sl}_2$). In this problem you will prove that any finite dimensional representation of $\mathfrak{sl}_2$ is a direct sum of irreducible representations, via the following steps:

The proof is by contradiction. Assume the contrary, so there exists some finite dimensional representation V that is not irreducible and also not a direct sum of smaller representations. By induction, let V be the representation of smallest dimension satisfying these assumptions.

(a) Show that the Casimir element C has only one eigenvalue on V, namely $\frac{n(n+2)}{2}$ for some nonnegative integer n.

(Hint: use the fact that the generalized eigenspace decomposition of C must be a decomposition of representations.)

(b) Show that V has a subrepresentation W isomorphic to V_n (where V_n is the irreducible representation from Problem 1) such that the representation V/W is isomorphic to the direct sum of m copies of V_n for some integer $m \geq 1$. (So $V/W \cong \bigoplus_{i=1}^{m} V_n$.)

(Hint: use (a) and the fact that V is the smallest reducible representation which cannot be decomposed.)

(c) Deduce from (b) that the weight space $V[n]$ (i.e., the eigenspace of $\rho(h)$ with eigenvalue n) is $m+1$ dimensional.

Pick a basis $v_1, \dots, v_{m+1}$ of $V[n]$. Show that $\{\rho(f)^j v_i\}_{\substack{i=1,\dots,m+1 \\ j=0,\dots,n}}$ are linearly independent and therefore form a basis of V.

(Hint: establish that if $v \in V[k]$ and $\rho(f)v = 0$ for $v \neq 0$, then $\rho(C)v = \frac{k(k-2)}{2} v$ and hence $k = -n$.)

(d) Define $W_i = \mathrm{span}(v_i, \rho(f)v_i, \dots, \rho(f)^n v_i)$ for $i = 1, \dots, m+1$. Show that W_i are subrepresentations of V and derive a contradiction to the fact that V cannot be decomposed.