18.704 PROBLEM SET 1
DUE: FEB. 26, 11:59PM EST

1. Show that the $\mathbb{C}$-vector space $\mathfrak{gl}_n$ consisting of all $n \times n$ matrices together with Lie bracket $[x, y] = x \cdot y - y \cdot x$ (where $x \cdot y$ means matrix multiplication) is a Lie algebra.

2. Let $\mathfrak{g} = \mathbb{C}$ with bracket $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

(a) Show that all irreducible representations of $\mathfrak{g}$ are 1-dimensional. Classify them.

(b) Classify the finite dimensional indecomposable representations of $\mathfrak{g}$.

(Hint: use Jordan normal form.)

3. Let $n \geq 0$ be a non-negative integer. Define $V^{(n)}$ to be the $\mathbb{C}$-vector space of homogeneous polynomials in two variables x, y of degree n. (So $V^{(0)} = \mathbb{C}$, $V^{(1)}$ has basis x, y, $V^{(2)}$ has basis x^2, xy, y^2 and so on.)

Let $\mathfrak{sl}_2$ act on $V^{(n)}$ as follows: for $P \in V^{(n)}$ considered as a function in x and y, let

$$h \cdot P := x\frac{\partial}{\partial x}(P) - y\frac{\partial}{\partial y}(P), \quad e \cdot P := x\frac{\partial}{\partial y}(P), \quad f \cdot P := y\frac{\partial}{\partial x}(P)$$

Show, using results from class, that any finite dimensional irreducible representation of $\mathfrak{sl}_2$ is isomorphic to $V^{(n)}$ for some $n \geq 0$.

4. Let $\mathfrak{g}$ be a Lie algebra. Then we can make $V = \mathfrak{g}$ a representation of $\mathfrak{g}$ by defining $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ as follows: for any $x \in \mathfrak{g}$ and $y \in V = \mathfrak{g}$, define

$$\rho(x)y := [x, y].$$

(a) Show that V is indeed a representation. (This is called the *adjoint representation* of $\mathfrak{g}$.)

(b) For $\mathfrak{g} = \mathfrak{sl}_2$, find the integer n such that the adjoint representation $V = \mathfrak{sl}_2$ is isomorphic to $V^{(n)}$. (In particular, $V = \mathfrak{sl}_2$ is irreducible.)