Spherical variety / L-function

F = ~~number~~ global field

$v,\ \mathcal{O}_v,\ F_v,\ \pi_v,\ q_v = \# k_v$

$\mathbb{A}$

Today: $G = PGL_2$, $H = \begin{pmatrix} * & \\ & 1 \end{pmatrix} = \mathbb{G}_m \hookrightarrow G$

$X = H\backslash G$ is example of affine G-spherical variety ($\Leftarrow$ normal, open B-orbit)

$T \subset B \subset G$ $\qquad X/B = H\backslash(G/B) = H\backslash\mathbb{P}^1 \qquad \mathbb{P}^1 = \{0\} \cup \mathbb{G}_m \cup \{\infty\}$

$[G] = G(F)\backslash G(\mathbb{A})$

~~$[H] \hookrightarrow [G]$~~

$\pi = \bigotimes_v{}' \pi_v \hookrightarrow L^2([G])_{cusp}$ irred cuspidal automorphic rep

π_v irrep of $G(F_v)$

$L(\pi, s) = L(\pi, std, s) = \prod_v L(\pi_v, s)$

Fact: Euler product converges for $Re(s) \gg 0$

~~Expect~~ Want: Meromorphic continuation & functional equation

Period integral $\qquad [H] \hookrightarrow [G]$

$$\int_{[H]} f(h)\, dh$$

$\psi : \mathbb{A}/F \to S^1$

Whittaker period: $\int_{[N]} f(n)\psi(n)\, dn = 1$

Thm (Hecke, Jacquet-Langlands) $\quad f \in C^\infty([G])$

Let f cuspidal eigenform, new form, normalized s.t. generating π

Then $\displaystyle\int_{[H]} f(h)|h|^{s-\frac{1}{2}}\, dh = \cancel{L(s,\pi)}\ L(\pi, s)$

In particular, $\displaystyle\int_{[H]} f(h)\, dh = L(\pi, \tfrac{1}{2}) = \prod_v L(\pi_v, \tfrac{1}{2})$ needs analytic continuation to make sense.

Conjecturally, can ~~attach~~ construct $L_X(\pi, s) \neq 0$ for X-distinguished $\pi \subset C^\infty([G])$ any affine spherical variety X

requires notion of dual gp $\check{G}_X$

(Gaitsgory-Nadler, Sakellaridis-Venkatesh, Knop)

(Ichino-Ikeda, Sakellaridis-Venkatesh)

Goal of seminar is to understand conjecture.

Back to $\mathbb{G}_m \backslash PGL_2$ $\qquad G = PGL_2$, $\check{G} = \check{G}_X = SL_2$

Reformulation of Hecke period in terms of Gross–Prasad periods

$\pi = \otimes'_v \pi_v$, $\quad f = \otimes'_v f_v$

Fix Hermitian forms $\langle\ ,\ \rangle : \pi_v \otimes \bar{\pi}_v \longrightarrow \mathbb{C}$ $\quad$ (π_v is unitary)

normalized s.t. $\langle f_v, f_v \rangle = 1$.

$$P^{Aut}(f,f) := \left| \int_{[H]} f \right|^2$$

lecke's Thm $\Rightarrow$ $P^{Aut}(f,f) = L(\pi, \tfrac{1}{2}) L(\bar{\pi}, \tfrac{1}{2}) \left| \int_{[N]} f(n)\psi(n)\, dn \right|^2$

GL_n

act (Rankin–Selberg theory, Jacquet, Ref: Sak-Venk, Lapid-Mao)

$$\left| \int_{[N]} f(n)\psi(n)\, dn \right|^2 = \prod_v \underbrace{\int_{N(F_v)} \langle \pi(n) f_v, f_v \rangle \psi(n)\, dn}$$

Almost all π_v are unramified, f_v is unramified (K_v-fixed) vector

$$= \frac{1}{L(\pi_v, Ad, 1)} \text{ at unramified } \pi_v$$

Jacquet-Langlands: $L(\pi_v, s) = \int_{H(F_v)} W_{f_v}(h)\, dh$ $\quad$ where $W_{f_v}(1) = 1$.

$$\Rightarrow \quad P^{Aut}(f,f) = \prod_v \underbrace{\int_{H(F_v)} \langle \pi(h) f_v, f_v \rangle\, dh} =: \prod_v P_v^{Planch}(f_v, f_v)$$

at π_v unramified,

$$P_v^{Planch}(f_v, f_v) = \frac{L(\pi_v, \tfrac{1}{2}) L(\bar{\pi}_v, \tfrac{1}{2})}{L(\pi_v, Ad, 1)} =: \frac{L_X(\pi_v)}{\cancel{L(\pi_v, Ad, 1)}}$$

hino-Ikeda conjecture says this is true (~~maybe~~ up to global constant)

for $H = SO_n \hookrightarrow G = SO_n \times SO_{n+1}$

or $n = 3$, $SO_2 = \mathbb{G}_m$, $SO_3 = PGL_2$, get back Hecke.

O. replaced by SU $\rightsquigarrow$ Thm of Wei Zhang.

For π_v unramified:

$\pi_v = \pi(\chi_v)$ principal series, $\quad \chi_v : T(F_v) \to \mathbb{C}^\times$ unram. character $\Leftrightarrow$ elt $\in \check{T}(\mathbb{C})$

$\quad\quad$ || lift of Satake parameter in $\check{T}(\mathbb{C})/W$

$$L_X(\chi_v) = L_X(\pi_v) = \frac{L(\pi_v, \tfrac{1}{2})\, L(\bar{\pi}_v, \tfrac{1}{2})}{L(\pi_v, Ad, 1)}$$

$$L_X(\chi_v) = L_X^{\frac{1}{2}}(\chi_v)\, L_X^{\frac{1}{2}}(\chi_v^{-1})$$

$$L_X^{\frac{1}{2}}(\chi_v) = \cancel{\frac{1 - q_v \cdot \chi_v(\check{\alpha}(\pi_v))}{(1 - \chi_v\, \check{\varepsilon}_1}} \quad \frac{1 - q_v\left(\chi_v(\pi_v^{\check{\alpha}})\right)}{\left(1 - \chi_v(\pi_v^{\check{\varepsilon}_1})\right)\left(1 - \chi_v(\pi_v^{-\check{\varepsilon}_2})\right)}$$

$\check{\varepsilon}_1(a) = \begin{pmatrix} a & \\ & 1 \end{pmatrix}$

$\check{\varepsilon}_2(a) = \begin{pmatrix} 1 & \\ & a \end{pmatrix}$

(for $G = PGL_2$, $\check{\varepsilon}_1 = -\check{\varepsilon}_2 = \frac{\check{\alpha}}{2}$).

N.B. $L_X^{\frac{1}{2}}(\chi_v)$ has half of $L(\pi_v, \tfrac{1}{2})$ and half of $L(\bar{\pi}_v, \tfrac{1}{2})$ $\quad\quad T = T_X$

$L_X^{\frac{1}{2}}(\chi_v)$ should be thought of as Mellin transform of a function on $T_X(F_v)/T_X(\mathcal{O}_v)$ $\;=\; \check{\Lambda}_T$

$$= \frac{1 - q_v \cdot 1\!\!1_{\check{\alpha}}}{(1 - 1\!\!1_{\check{\varepsilon}_1})(1 - 1\!\!1_{-\check{\varepsilon}_2})}(\chi_v)$$ where $1\!\!1_{\check{\alpha}}$ means indicator function at $\check{\alpha} \in \check{\Lambda}_T$

$$= \frac{1 - e^{\check{\alpha}}}{\left(1 - q^{-\frac{1}{2}} e^{\check{\varepsilon}_1}\right)\left(1 - q^{-\frac{1}{2}} e^{-\check{\varepsilon}_2}\right)}(\chi_v)$$ where $e^{\check{\lambda}} := q_v^{\langle \check{\lambda}, \rho \rangle}\, 1\!\!1_{\check{\lambda}}$

$L^{\frac{1}{2}}(\chi_v)$ is unramified Plancherel measure on $L^2(X(F_v))$

$$= \frac{(\text{numerator})}{(\text{denominator})}$$

in general, numerator is product of positive spherical roots of $\check{G}_X$

denominator is combinatorial: B-divisors of X (colors) and W_X-action

(mysterious)

• should also be describable using resolution of cotangent bundle of X.

Bernstein morphism / Asymptotics $\quad\quad X = X(F_v)$, $X_\emptyset = X_\emptyset(F_v)$

$e_\emptyset^* : C^\infty(X) \longrightarrow C^\infty(X_\emptyset)$

Φ^0 basic function

Calculating $L^{\frac{1}{2}}(\chi_v) \Longleftrightarrow$ calculating $e_\emptyset^*(\Phi^0)$