Formulation of Global Conjecture of Sakellaridis–Venkatesh

3/6/19

$X = H\backslash G$ homogeneous spherical (affine $\iff$ reductive) by Matsushima's Criterion
(" = normal, open Borel orbit)

[Lu73] Luna, Slices étales, dimo
[Ric77] Richardson, Affine coset spaces of reductive alg gps.

F = global field

$\mathcal{L}_F$ = (conjectural) Langlands group

Def Global <u>Arthur parameter</u>: $\psi : \mathcal{L}_F \times SL_2(\mathbb{C}) \to \check{G}$

- $\mathcal{L}_F$ has bounded image
- SL_2 restriction is algebraic

$$\mathcal{L}_F \longrightarrow \mathcal{L}_F \times SL_2(\mathbb{C}) \xrightarrow{\psi} \check{G} \quad \text{gives Langlands parameter}$$
$$w \longmapsto \left(w, \begin{pmatrix} |w|^{\frac12} & \\ & |w|^{-\frac12} \end{pmatrix}\right)$$

$\check{G}$-conj class $[\psi] \rightsquigarrow$ ~~A packet of ψ~~ ~~finite set of~~

$$\mathcal{A}_{[\psi]} \subset L^2([G]) \qquad [G] = G(F)\backslash G(\mathbb{A})$$

subspace of automorphic forms

Arthur conjecture: $L^2([G]) = \int \mathcal{A}_{[\psi]}\, \mu(\psi)$

(local-global compatibility)

Defn X-distinguished Arthur parameter is commutative diagram:

$$\begin{tikzcd} & \check{G}_X \times SL_2 \arrow[dr] & \\ \mathcal{L}_F \times SL_2 \arrow[ur, "\phi \times id"] \arrow[rr] & & \check{G} \end{tikzcd}$$

to be constructed

where ϕ is tempered (bounded on $\mathcal{L}_F$) Langlands parameter into $\check{G}_X$.

Rmk X-distinguished Arthur parameter is Arthur parameter into $\check{G}$ plus a lift

Assumptions: π irreducible unitary rep of $G(\mathbb{A})$ tempered embedding $\hookrightarrow C^\infty([G])$
(π cuspidal for simplicity)

$\pi = \bigotimes_v' \pi_v$

- Multiplicity one: $\dim \operatorname{Hom}_{G(F_v)}(\pi_v, C^\infty(X(F_v))) \leq 1$
  (Jacquet shows mult one condition too restrictive)
- Use Tamagawa measure everywhere.

$$\mathcal{P}^{Aut}(f_1, f_2) = \int_{[H]} f_1\, dh \cdot \int_{[H]} \overline{f_2}\, dh \qquad \text{Hermitian}$$

$\langle\ ,\ \rangle : \pi \otimes \overline{\pi} \longrightarrow \mathbb{C}$

$\mathcal{P}_v^{Planch} : \pi_v \otimes \overline{\pi}_v \longrightarrow \mathbb{C}$ $\quad H(F_v)$-biinvariant Hermitian

$$\mathcal{P}_v^{Planch}(u_1, u_2) = \int_{H(F_v)} \langle \pi_v(h) u_1, u_2 \rangle\, dh$$

(RHS need strongly tempered assumption on X, G but $\mathcal{P}_v^{Planch}$ can be defined even without it)

Global (Period) Conjecture

Let ψ an X-distinguished Arthur-parameter (assume $\check{G}_X \subset \check{G}$) [multiplicity one $\Downarrow$]

Choose $\mathcal{A}'_{[\psi]} \subset \mathcal{A}_{[\psi]}$ to contain each irrep w/ mult. one and $\mathcal{P}^{Aut}$ is zero on orthogonal complement of $\mathcal{A}'_{[\psi]}$. ($\mathcal{A}'_{[\psi]}$ exists by multiplicity one assumption on X)

$$\boxed{\mathcal{P}^{Aut}\big|_{\pi} = \mathbb{Q}^{\times} \cdot \prod_v{}' \mathcal{P}_v^{Planch}}$$

For $\pi \hookrightarrow \mathcal{A}'_{[\psi]}$.

RHS doesn't converge: for π_v unramified, $u_0 \in \pi_v^{K_v}$, $\langle u_0, u_0 \rangle = 1$

$$\frac{\mathcal{P}_v^{Planch}(u_0, u_0)}{\langle u_0, u_0 \rangle} = \frac{L_{X,v}(\text{central value})}{\text{"Plancherel measure for } G_X\text{"}}$$

Sakellaridis, "Spherical functions on Spherical Varieties"

G_X is dual group of $\check{G}_X$.

Part of conjecture is that $\prod_v L_{X,v}(s)$ has analytic continuation. and we can make sense of $\prod_v{}' \mathcal{P}_v^{Planch}$ using analytic cont.

Spectrally, $L_X(s)$ seems to be L-function for some repn of $\underline{\underline{\check{G}_X}}$

E.g. (Godement – Jacquet, Tate)

$X = \mathrm{Mat}_n \circlearrowleft G = GL_n \times GL_n$ $\qquad L(\pi, \overset{GL_n}{\circlearrowright}\mathrm{Std}, s)$

(Rankin–Selberg)

$X = \overline{P_n^{diag} \backslash GL_n \times GL_n}^{\,aff}$ $\qquad L(\pi_1 \times \pi_2, \otimes, s)$

$\uparrow$ not affine

$P_n = \begin{pmatrix} * & * \\ & 1 \end{pmatrix}$ mirabolic subgroup