

Formulation of Global Conjecture of Sakellaridis-Venkatesh

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$X = H \backslash G$ homogeneous spherical (affine \leftrightarrow reductive)

$F =$ global field

(normal, open Borel orbit)

by Matsushima's Criterion

[Lu73] Luna, Slices étales, etc.

[Ric77] Richardson, Affine coset spaces of reductive alg. gps.

$\mathcal{L}_F =$ (conjectural) Langlands group

Def Global Arthur parameter: $\psi: \mathcal{L}_F \times SL_2(\mathbb{C}) \rightarrow \check{G}$

• \mathcal{L}_F has bounded image

• SL_2 restriction is algebraic

$\mathcal{L}_F \rightarrow \mathcal{L}_F \times SL_2(\mathbb{C}) \xrightarrow{\psi} \check{G}$ gives Langlands parameter

$$w \mapsto \begin{pmatrix} w & |w|^{\frac{1}{2}} \\ & |w|^{-\frac{1}{2}} \end{pmatrix}$$

\check{G} -conj class $[\psi] \rightsquigarrow$ ~~A parameter of ψ finite set of~~

$$\mathcal{A}_{[\psi]} \subset L^2([G])$$

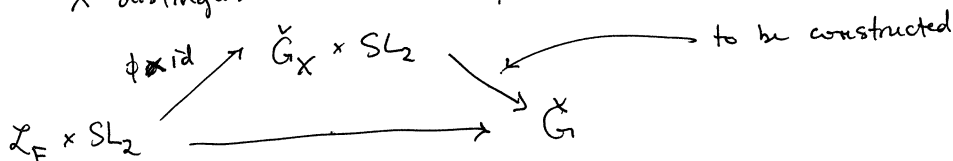
$$[G] = G(F) \backslash G(\mathbb{A})$$

subspace of automorphic forms

Arthur conjecture: $L^2([G]) = \int \mathcal{A}_{[\psi]} \mu(\psi)$

(local-global compatibility)

Defn X -distinguished Arthur parameter is commutative diagram:



where ϕ is tempered (bounded on \mathcal{L}_F) Langlands parameter into \check{G}_X .

Rank X -distinguished Arthur parameter is Arthur parameter into \check{G} plus a lift

Assumptions: (π cuspidal for simplicity)
 π irreducible unitary rep of $G(\mathbb{A})$ tempered embedding $\hookrightarrow C^\infty([G])$

$$\pi = \otimes'_v \pi_v$$

• Multiplicity one: $\dim \text{Hom}_{G(F_v)}(\pi_v, C^\infty(X(F_v))) \leq 1$

(Jacquet shows mult one condition too restrictive)

• Use Tamagawa measure everywhere.

$$\mathcal{P}^{\text{Aut}}(f_1, f_2) = \int_{[H]} f_1 dh \cdot \int_{[H]} \overline{f_2} dh \quad \text{Hermitian}$$

$$\langle \cdot, \cdot \rangle : \pi \otimes \overline{\pi} \rightarrow \mathbb{C}$$

$$\mathcal{P}_v^{\text{Planch}} : \pi_v \otimes \overline{\pi}_v \rightarrow \mathbb{C} \quad H(F_v)\text{-bimvariant Hermitian}$$

$$\mathcal{P}_v^{\text{Planch}}(u_1, u_2) = \int_{H(F_v)} \langle \pi_v(h) u_1, u_2 \rangle dh$$

(RHS need strongly tempered assumption on X, G
but $\mathcal{P}_v^{\text{Planch}}$ can be defined even without it)

Global Conjecture (Period)

Let ψ an X -distinguished Arthur-parameter (assume $\check{G}_X \subset \check{G}$)

Choose $\mathcal{A}'_{[\psi]} \subset \mathcal{A}_{[\psi]}$ to contain each irrep w/ mult. one

and \mathcal{P}^{Aut} is zero on orthogonal complement of $\mathcal{A}'_{[\psi]}$. ($\mathcal{A}'_{[\psi]}$ exists by multiplicity one assumption on X)

$$\boxed{\mathcal{P}^{\text{Aut}}|_{\pi} = \mathbb{Q}^{\times} \cdot \prod_v \mathcal{P}_v^{\text{Planch}}}$$

For $\pi \hookrightarrow \mathcal{A}'_{[\psi]}$.

RHS doesn't converge: for π_v unramified, $u_0 \in \pi_v^{K_v}$, $\langle u_0, u_0 \rangle = 1$

$$\frac{\mathcal{P}_v^{\text{Planch}}(u_0, u_0)}{\langle u_0, u_0 \rangle} = \frac{L_{X,v}(\text{central value})}{\text{"Plancherel measure for } \check{G}_X \text{"}} \quad \text{Sakellaridis, "Spherical functions on Spherical Varieties"}$$

G_X is dual group of \check{G}_X .

Part of conjecture is that $\prod_v L_{X,v}(s)$ has analytic continuation, and we can make sense of $\prod_v \mathcal{P}_v^{\text{Planch}}$ using analytic cont.

Spectrally, $L_X(s)$ seems to be L-function for some repn of \check{G}_X

E.g. (Godement-Jacquet, Tate)

$$X = \text{Mat}_n \hookrightarrow G = \text{GL}_n \times \text{GL}_n \quad L(\pi, \overset{\text{GL}_n}{\downarrow} \text{Std}, s)$$

(Rankin-Selberg)

$$X = \frac{\text{P}_n^{\text{diag}}}{n} \backslash \text{GL}_n \times \text{GL}_n \xrightarrow{\text{aff}}$$

$$L(\pi_1 \times \pi_2, \otimes, s)$$

\uparrow
not affine

$$\text{P}_n = \left(\begin{array}{c|c} * & * \\ \hline * & 1 \end{array} \right) \text{ mirabolic subgroup}$$