SPHERICAL VARIETIES, L-FUNCTIONS, AND CRYSTAL BASES

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Abstract. The program of Sakellaridis and Venkatesh proposes a unified framework to study integral representations of L-functions through the lens of spherical varieties. For X an affine spherical variety, the (hypothetical) IC complex of the infinite-dimensional formal arc space of X is conjecturally related to special values of local unramified L-functions. We formulate this relation precisely using a new conjectural geometric construction of the crystal basis of a finite-dimensional representation (determined by X) of the dual group. We prove these conjectures for a large class of spherical varieties. This is joint work with Yiannis Sakellaridis.

Contents

Let $\mathbb{F}_q$ be a finite field, $k = \overline{\mathbb{F}}_q$, $F = \mathbb{F}_q(\!(t)\!)$ the local field and $O = \mathbb{F}_q[\![t]\!]$ the ring of integers. (After the introduction I will replace $\mathbb{F}$ with k everywhere while keeping the same notation. One can also take $k = \mathbb{C}$.) Let G be a connected reductive group over $\mathbb{F}$. For simplicity in this talk we assume G is split (but our results hold without it).

1. Spherical varieties

1.1. **What is a spherical variety?** A G-variety X over $\mathbb{F}_q$ is called *spherical* if X_k is a normal variety with an open dense orbit of a Borel $B_k \subset G_k$ after base change to k.

One should think of this as a finiteness property. For example, Brion proved the above definition is equivalent to X_k having finitely many B_k orbits. The point is that spherical varieties have good combinatorics: they have now been classified (over $\mathbb{C}$) in a way analogous to the classification of split reductive groups via root datum. If you want to cross the bridge of Langlands duality, you will need to use a little combinatorics at some point.

The following also hold if k has characteristic 0.

Theorem 1 ([VK])**.** *An affine variety X is spherical if and only if $k[X]$ has multiplicity one as an algebraic G_k-representation.*

Theorem 2 ([SV, Theorem 5.1.5])**.** *If $X^\bullet$ is a quasi-affine spherical variety satisfying the* wavefront *assumption, then $\mathrm{Hom}_{G(F)}(\pi, C^\infty(X^\bullet(F)))$ is finite dimensional for all smooth irreducible $G(F)$-representations π.*

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Most results for spherical varieties are in characteristic 0 as Frobenius introduces some new cases in characteristic p. If you assume existence of an integral model for X then everything holds for p large enough.

Example 1. I'll give more explicit examples later, but for now some examples of spherical varieties are:

- toric varieties (the same definition with $G = T$ a torus)
- symmetric spaces $K\backslash G$, which have been studied extensively in representation theory and number theory
  - As a particular case of a symmetric space, consider $X = G'$ a reductive group, acted on by $G = G' \times G'$ via left and right multiplication. We will refer to this as the *group case*.

1.2. **Why are they relevant?** We give a simplified, imprecise summary of the local conjectures of Sakellaridis [Sak12], further developed in joint work with Venkatesh [SV]. For a much better succinct summary of the conjectures of Sakellaridis–Venkatesh, see the introduction of [GW].

Conjecture 1 (Sakellaridis, Sakellaridis–Venkatesh)**.** *For any* affine spherical G*-variety* X [1]*, and an irreducible unitary* $G(F)$*-representation* π*, there is an "integral" (more precisely,* $|\mathcal{P}_X|^2_\pi$ *is really a Hermitian pairing* $\pi \otimes \overline{\pi} \to \mathbb{C}$ *but let's pretend it's just a number given by some integral)*

$$|\mathcal{P}_X|^2_\pi$$

involving the IC function of $X(O)$ *such that*

(i) $|\mathcal{P}_X|^2_\pi \neq 0$ *determines a functorial lifting of* π *to* $\sigma \in \mathrm{Irr}(G_X(F))$ *corresponding to a map* $\check{G}_X(\mathbb{C}) \to \check{G}(\mathbb{C})$*. Meaning: if* π *corresponds under Local Langlands to a homomorphism* $W_F \to \check{G}(\mathbb{C})$ *where* W_F *is the Weil group of* F*, then there exists a lifting*

$$\begin{array}{ccc} & & W_F \\ & \overset{\exists}{\swarrow} & \downarrow \\ \check{G}_X(\mathbb{C}) & \longrightarrow & \check{G}(\mathbb{C}) \end{array}$$

such that the lift corresponds under Local Langlands to σ*. (The conjecture is formulated more precisely using Vogan's Arthur packets, but I omit these subtleties.)*

(ii) *there should exist a* $\check{G}_X$*-representation*

$$\rho_X : \check{G}_X(\mathbb{C}) \to \mathrm{GL}(V_X)$$

such that $|\mathcal{P}_X|^2_\pi = L(\sigma, \rho_X, s_0)$ *for a special value* s_0 *[up to known constants and zeta factors].*

In this talk I will focus more on (ii) and soon we will just assume $\check{G}_X = \check{G}$ to avoid subtleties related to (i). But before that let me mention that the map $\check{G}_X \to \check{G}$ has been constructed (see below for history) so it is a known entity, whereas the representation ρ_X is very mysterious and *a priori* the ρ_X are only determined on the basis of examples from numerical calculations. One of the products of our work is that we give a formula for what ρ_X has to be, solely in terms of the prime B-divisors of X.

[1] I am taking liberties in the statement, there are extra assumptions and nothing should be taken literally.

1.3. **History of $\check{G}_X$.** This is largely for educational purposes as I will later assume $\check{G}_X = \check{G}$, but let me describe the history behind the spherical dual group. The goal is to construct a map

$$\check{G}_X \to \check{G}$$

with finite kernel.

- the dual maximal torus $\check{T}_X$ is easy to define
- the Weyl group W_X of $\check{G}_X$ and the root system of spherical roots
  - was known for symmetric varieties very early on (Cartan '27); here W_X is called the little Weyl group, spherical root system is the restricted root system;
  - for a spherical variety, Brion ('90) showed existence of W_X as a finite reflection group of a fundamental domain using previous work of Luna–Vust ('83). He also showed existence of a root system of spherical roots.
  - Knop ('90, '93, '94) then defined the Weyl group for any irreducible G-variety in several different ways and showed they were all equivalent (and also equivalent to Brion's definition). He used the moment map $T^*X \to \mathfrak{g}^*$ and separately, invariant differential operators $\mathcal{D}(X)^G$.
- Independent from work of Brion, Knop, Gaitsgory–Nadler [GN] define a subgroup $\check{G}_X^{GN} \subset \check{G}$ using Tannakian formalism, but they don't show its Weyl group coincides with Brion's
- You might think that if you have W_X and a root system, you already have $\check{G}_X$, but there is an issue of integrality: you need the coroots to lie in the lattice corresponding to $\check{T}_X$. Sakellaridis–Venkatesh [SV] suggested a way to normalize the spherical roots such that now they can define $\check{G}_X$ combinatorially. But the story is not over: you really want a distinguished map $\check{G}_X \to \check{G}$ (conjecturally with image $\check{G}_X^{GN}$). They construct this map under assumptions about [GN] (which are still unchecked today).
- Knop–Schalke [KS]: define the map $\check{G}_X \to \check{G}$ combinatorially unconditionally.

1.4. For the purposes of this talk, you can pick an example out of this table:

In Table 1, $T^*V' = V' \oplus V'^*$. The names signify who discovered the corresponding integrals

TABLE 1. Langlands dual data

	$X \circlearrowleft G$	$\check{G}_X$	V_X
Usual Langlands	Group $G' \circlearrowleft G' \times G' = G$	$\check{G}'$	$\check{\mathfrak{g}}'$
Whittaker normalization	$(N, \psi)\backslash G$	$\check{G}$	pt
Tate's thesis	$\mathbb{A}^1 \circlearrowleft \mathbb{G}_m$	$\mathbb{G}_m$	$T^*\mathbb{C}$
Hecke	$\mathbb{G}_m\backslash \mathrm{PGL}_2$	$\check{G} = \mathrm{SL}_2$	$T^*\mathrm{std}$
Rankin–Selberg, Jacquet–Piatetski-Shapiro–Shalika	$\mathrm{GL}_n \times \mathbb{A}^n \circlearrowleft \mathrm{GL}_n \times \mathrm{GL}_n$, $H =$ diagonal mirabolic	$\check{G}$	$T^*(\mathrm{std} \otimes \mathrm{std})$
loc cit.	$\mathrm{GL}_n\backslash \mathrm{GL}_{n+1} \times \mathrm{GL}_n$	$\check{G}$	$T^*(\mathrm{std} \otimes \mathrm{std})$
Gan–Gross–Prasad	$\mathrm{SO}_{2n}\backslash \mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n}$	$\check{G} = \mathrm{SO}_{2n} \times \mathrm{Sp}_{2n}$	$\mathrm{std} \otimes \mathrm{std}$
Jacquet, Ichino	$\mathrm{PGL}_2^{\mathrm{diag}}\backslash \mathrm{PGL}_2^{\times 3}$	$\check{G} = \mathrm{SL}_2^{\times 3}$	$\mathrm{std} \otimes \mathrm{std} \otimes \mathrm{std}$
	Example 2	$\check{G} = \mathrm{GL}_2^{\times n} \times \mathbb{G}_m$	$T^*(\mathrm{std}_2^{\otimes n} \otimes \mathrm{std}_1)$

and determined what V_X should be. Of course in these cases the named people have discovered far more about each case than what I will discuss today.

But now you can use spherical varieties to try to find new examples people haven't discovered before:

Example 2 ([Sak12, §4.5])**.** A new family of examples is provided by Sakellaridis generalizing the Rankin–Selberg convolution to an integral representation of the n-fold tensor product L-function for GL_2. Let $G = \mathrm{GL}_2^{\times n} \times \mathbb{G}_m$ acting on $X^\bullet = H\backslash G$ where

$$H = \left\{ \begin{pmatrix} a & x_1 \\ & 1 \end{pmatrix} \times \begin{pmatrix} a & x_2 \\ & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} a & x_n \\ & 1 \end{pmatrix} \times a \,\middle|\, x_1 + x_2 + \cdots + x_n = 0 \right\}.$$

Let X be the affine closure of $X^\bullet$. In this case $\check{G}_X = \check{G} = \mathrm{GL}_2^{\times n} \times \mathbb{G}_m$, and it will follow from our work that $V_X = T^*(\mathrm{std}_2^{\otimes n} \otimes \mathrm{std}_1)$ (and there is an integral representation of the corresponding L-function).

For $n = 3$ this coincides with a construction of Garrett, worked on by many people.

The slogan is if we want to find more unknown examples, we need to look at singular X, necessarily not equal to $H\backslash G$.

Theorem 3 ([Lun73], [Ric77])**.** *The variety $H\backslash G$ is affine if and only if H is reductive*

1.5. **Assumption $\check{G}_X = \check{G}$.** Note that in Table 1, in all but the first row $\check{G}_X = \check{G}$. This is the situation that I will restrict to today. The point of the talk will be that for our results, this assumption allows us to reduce everything to the Hecke case of $\mathbb{G}_m\backslash\mathrm{PGL}_2$.

So far I haven't told you anything about $\check{G}_X$ besides some history, so how are you supposed to interpret this assumption?

Assumption 1. For this talk, assume $\check{G}_X = \check{G}$ and X has no type N roots[2].

This is equivalent to the following (after base change to k):

- X has an open B-orbit X° acted on simply transitively by B (so after choose a base point $x_0 \in X^\circ$ we get $X^\circ \cong B$),
- $X^\circ P_\alpha/\mathcal{R}(P_\alpha) \cong \mathbb{G}_m\backslash\mathrm{PGL}_2$ for every simple α. Here $P_\alpha \supset B$ is the standard sub-minimal parabolic corresponding to α.

So this says X has open subvarieties which "look" like the Hecke case, and the complement of these opens are certain B-divisors.

2. Function-theoretic results

2.1. **Sakellaridis–Venkatesh á la Bernstein.** The most conceptually satisfactory way to explain how to get an L-function from X is through a lengthy discussion on Plancherel decomposition for $L^2(X(F))$. However this takes a long time, so for brevity I will go with the fastest way instead. We refer to [SW, Introduction] for details.

Sakellaridis–Venkatesh [SV] developed the theory of Bernstein [Ber] of *asymptotics* to study Plancherel decomposition. Skipping all intermediate steps, they show that a key computation for studying the *unramified* spectrum is to consider the operator

$$\pi_! : C_c^\infty(X(F))^{G(O)} \to C^\infty(N(F)\backslash G(F))^{G(O)}$$

[2] 'N' is for normalizer. We want to avoid examples like $\mathrm{O}_n\backslash\mathrm{GL}_n$, which Jacquet, Mao have shown has some metaplectic behavior which is not expected to be related to L-functions

defined by

$$\pi_! \Phi(g) := \int_{N(F)} \Phi(x_0 n g) dn, \quad g \in G(F)$$

where $x_0 \in X^\circ(\mathbb{F}_q)$ is a fixed base point in the open B-orbit. This is an integral over generic horocycles, so we call $\pi_!$ the *X-Radon transform.*

Note that $\pi_! \Phi$ is a function on $N(F)\backslash G(F)/G(O) = T(F)/T(O) = \check{\Lambda}$.

For those more familiar with harmonic analysis, you can believe that the X-Radon transform is related to finding formulas for spherical functions (i.e., unramified Hecke eigenfunctions) on $X(F)$. And as already mentioned, Bernstein asymptotics relates the Radon transform to the unramified Plancherel measure of $X(F)$.

2.2. **Conjecture on Radon transform.** The conjecture can be made for any $\check{G}_X$ but it is more awkward to state, so for precision I will only state the case $\check{G}_X = \check{G}$:

Conjecture 2. *Assume $\check{G}_X = \check{G}$ and X has no type N roots. Let Φ_0 denote the IC function of $X(O)$. Then there exists a symplectic $V_X \in \mathrm{Rep}(\check{G})$ with a $\check{T}$ polarization $V_X = V_X^+ \oplus (V_X^+)^*$ such that*

$$\pi_! \Phi_0 = \frac{\prod_{\check{\alpha} \in \check{\Phi}_G^+} (1 - q^{-1} e^{\check{\alpha}})}{\prod_{\check{\lambda} \in \mathrm{wt}(V_X^+)} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})} \in \mathrm{Fn}(\check{\Lambda}) \tag{2.1}$$

where $e^{\check{\lambda}}$ is the indicator function of $\check{\lambda}$ and $e^{\check{\lambda}} e^{\check{\mu}} = e^{\check{\lambda} + \check{\mu}}$

The fact that V_X is supposed to be *symplectic* is special to the $\check{G}_X = \check{G}$ case.

The Euler product on the right should be understood via a power series expansion:

$$\frac{1}{1 - q^{-\frac{1}{2}} e^{\check{\lambda}}} = \sum_{n \geq 0} (q^{-\frac{1}{2}} e^{\check{\lambda}})^n.$$

What (2.1) is really saying is that $\pi_! \Phi_0$ is supposed to give "half" of an L-function. More specifically, you can take the *Mellin transform* of any function on $T(F)/T(O)$ to get a function on $\check{T}(\mathbb{C})$ (ignoring convergence issues). In practice, this means for $\chi \in \check{T}(\mathbb{C})$, replace $e^{\check{\lambda}}$ in the above formula by $\check{\lambda}(\chi)$, where $\check{\lambda}$ is considered as a weight of $\check{T}(\mathbb{C})$. Then the Mellin transform of $\pi_! \Phi_0$ is

$$\widehat{\pi_! \Phi_0}(\chi) = \frac{L(\chi, V_X^+, \frac{1}{2})}{L(\chi, \check{\mathfrak{n}}, 1)}, \text{ this is "half" of } \frac{L(\chi, V_X, \frac{1}{2})}{L(\chi, \check{\mathfrak{g}}/\check{\mathfrak{t}}, 1)}$$

Note that $L(\chi, \check{\mathfrak{t}}, 1)$ is a product of zeta functions which do not depend on χ, so they are normalized out.

The special value at the *central value* $1/2$ is specific to the $\check{G}_X = \check{G}$ case. In some sense, this makes the $\check{G}_X = \check{G}$ case the most interesting to study.

2.3. **Previous work.** When $X = H\backslash G$ and H is reductive (equivalent to $H\backslash G$ being affine), Sakellaridis ([Sak08, Sak13]) proved the above conjecture (without restriction on $\check{G}_X$) using function-theoretic techniques. So if you're only interested in these cases at the function-theoretic level, we have nothing new to offer (although we give a different geometric proof in this case as well). On the other hand, he does not consider affine embeddings $X \supsetneq H\backslash G$, which we do under the $\check{G}_X = \check{G}$ assumption. In this case X is smooth so Φ_0 is just the indicator function of $X(O)$.

However when X is singular, geometric considerations must be made to understand Φ_0 since IC is in the very definition.

Theorem 4. *Explicit formula for (2.1) has been established using geometric techniques in the following cases:*

- *Braverman–Finkelberg–Gaitsgory–Mirković [BFGM]:*
  - $X = \overline{N^-\backslash G}$, $\check{G}_X = \check{T}$, $V_X = \check{\mathfrak{g}}^*/\check{\mathfrak{t}}^*$. *Note that* $\check{G} \times^{\check{G}_X} V_X = T^*(\check{G}/\check{T})$, *which has some global incarnation as geometric Eisenstein series*
- *Bouthier–Ngô–Sakellaridis [BNS]:*
  - X *toric variety,* $G = T$, $\check{G}_X = \check{T}$, *weights of* V_X *correspond to generators of the monoid equal to* $\mathrm{Hom}_{monoid}(\mathbb{G}_a, X)$.
  - $X \supset G'$ *is an L-monoid, so here the group is* $G = G' \times G'$, $\check{G}_X = \check{G}'$, *and* $V_X = \check{\mathfrak{g}}' \oplus T^*V^{\check{\lambda}}$ *where* $\check{\lambda}$ *is the coweight appearing in the definition of an L-monoid.*

In these geometric cases $\check{G}_X \neq \check{G}$. Our result is:

Theorem 5 (Sakellaridis–W)**.** *Assume* X *affine spherical,* $\check{G}_X = \check{G}$ *and* X *has no type N roots*[3]*. Then*

$$\pi_! \Phi_{\mathrm{IC}_{X(O)}} = \frac{\prod_{\check{\alpha} \in \check{\Phi}_G^+}(1 - q^{-1} e^{\check{\alpha}})}{\prod_{\check{\lambda} \in \mathrm{wt}(V_X^+)}(1 - q^{-\frac{1}{2}} e^{\check{\lambda}})}$$

for some $V_X^+ \in \mathrm{Rep}(\check{T})$ *such that:*

(i) $V'_X := V_X^+ \oplus (V_X^+)^*$ *has action of* $(\mathrm{SL}_2)_\alpha$ *for every simple root* α
  - *We do not check the Weyl/Serre relations, which would imply* V'_X *is a* $\check{G}$*-representation.*

(ii) *Assuming* V'_X *satisfies Weyl/Serre relations (so it is a* $\check{G}$*-representation), we determine its highest weights with multiplicities (in terms of prime* B*-divisors of* X*).*

Remarks: (ii) gives a recipe for the previously mysterious conjectural V_X in terms of X. Namely, set V_X equal to the direct sum of the highest weight representations corresponding to the highest weights from (ii), which can be defined just using data from X. We are saying this is what V_X has to be for Conjecture 2 to be true.

It is a consequence of (i) that if this newly defined V_X is a minuscule representation, then we must have $V_X = V'_X$, so we have proved Conjecture 2 in this case.

We also showed that:

Proposition 1. *If* $X = H\backslash G$ *with* H *reductive, then* V_X *is minuscule.*

This explains why mostly minuscule cases have appeared so far.

2.3.1. We can reduce the checking of Weyl relations (which imply Serre relations for finite dimensional representations) to the cases where $X = \overline{H\backslash G}$ and G has semisimple rank 2. Now if one looks at Wasserman's tables of rank 2 spherical varieties, there are only around 10 that satisfy $\check{G}_X = \check{G}$ (the only exceptional group is G_2, of which there are 3 cases). This didn't seem easy to check, but it also does not seem impossible.

2.3.2. There is some hope that our techniques will generalize to any X (no restriction on $\check{G}_X$) by combining the knowledge from [BFGM, BNS].

3. Geometry

From now on I will base change to k while keeping the same notation. (One can also take $k = \mathbb{C}$ now.) Let $\mathbf{X_O}$ denote the formal arc space of X, so $\mathbf{X_O}(k) = X(k[\![t]\!])$.

Recall we will assume Assumption 1 throughout. So we fix a base point $x_0 \in X^\circ(k)$ and identify $X^\circ \cong B$. Let $H \subset G$ be the stabilizer of x_0, so $HB \subset G$ is open dense.

[3] Technically we need some further assumptions over $\mathbb{F}_q$ to ensure X behaves like it does in characteristic 0

3.0.1. First problem: $\mathbf{X_O}$ is an affine scheme of infinite type, and there is currently no theory of perverse sheaves on such spaces (although at least in our setup it's expected there should be such a theory). Nevertheless, Bouthier–Ngo–Sakellaridis [BNS] show that the IC function of $\mathbf{X_O}$, which should equal the trace of geometric Frobenius of $\mathrm{IC}_{\mathbf{X_O}}$, is well-defined. They use a theorem of Grinberg–Kazhdan (characteristic 0) and Drinfeld (any characteristic):

Theorem 6 (Grinberg–Kazhdan, Drinfeld)**.** *Let $\gamma \in X(k[\![t]\!])$ be an arc that generically lands in the smooth locus of X. Then there exists a finite type scheme Y and $y \in Y(k)$ such that there is an isomorphism of formal neighborhoods*

$$(\widehat{\mathbf{X_O}})_\gamma \cong \widehat{Y}_y \times \widehat{\mathbb{A}}^\infty.$$

I.e., near generic arcs $\mathbf{X_O}$ has finite-type singularities.

We call Y as above a *model* of $\mathbf{X_O}$.

3.1. **Zastava space.** We will use the fact that Drinfeld's proof [Dri18] of this theorem gives us explicit models for $\mathbf{X_O}$. This phenomenon was first used by Finkelberg–Mirković to study $X = \overline{G/N}$ ($\check{G}_X = \check{T}$). The two models are:

(i) the *Zastava space*[4] $\mathcal{Y}_X = \mathrm{Maps}_{\mathrm{gen}}(C, X/B \supset X^\circ/B)$

(ii) the Artin stack $\mathcal{M}_X = \mathrm{Maps}_{\mathrm{gen}}(C, X/G \supset X^\bullet/G)$.

I will downplay the role of $\mathcal{M}_X$ in this talk, but it is very important for modeling the Hecke action of $G(F)$ on $X^\bullet(F)$.

3.1.1. Our assumptions imply that the stack X/B contains $X^\circ/B = \mathsf{pt}$ as an open substack.

A point $y \in \mathcal{Y}_X(k)$ is a map $C \to X/B$ generically landing in pt. So by Beauville–Laszlo's theorem

$$y \leftrightarrow \left\{\begin{matrix} \text{finite set } \{v_i\}_{i\in I} \subset C(k), \\ \hat{y}_i \in (X(O_{v_i}) \cap X^\circ(F_{v_i}))/B(O_{v_i}), \\ y(C - \{v_i\}) = \mathsf{pt} \end{matrix}\right\}$$

Recall we are using $x_0 \in X^\circ(k)$ to identify $X^\circ \cong B$. Then

$$X^\circ(F_{v_i})/B(O_{v_i}) \cong \mathbf{B}_{\mathbf{F}_{v_i}}/\mathbf{B}_{\mathbf{O}_{v_i}}(k) = \mathrm{Gr}_{B,v_i}(k)$$

Now recall that Gr_B has the same connected components as Gr_T, which are indexed by the coweight lattice $\check{\Lambda}$. So to each $\hat{y}_i$ is attached a coweight $\check{\lambda}_i \in \check{\Lambda}$.

From this we see that $\mathcal{Y}$ lives over a space

$$\left\{ \check{\Lambda}\text{-valued divisors} : \sum_{i\in I} \check{\lambda}_i \cdot v_i,\ v_i \in C(k) \text{ distinct} \right\}$$

If $\check{\lambda}_i$ could be any coweight then we would need something fancy like the Ran space to make sense of the above. However, since $\hat{y}_i \in X(O_{v_i})$ is an arc, all the $\check{\lambda}_i$ belong to a strictly convex cone. So there is a sense of "positive" grading. More specifically,

$$\pi : \mathcal{Y} \to \mathcal{A} = \mathrm{Maps}(C, X/\!\!/N/T).$$

Let me assume for ultimate simplicity that $X/\!\!/N = \mathbb{A}^r \supset \mathbb{G}_m^r = T$ with a corresponding basis $\check{\nu}_1, \ldots, \check{\nu}_r \in \check{\Lambda}$ for the cocharacters whose limit as $t \to 0$ lands in $X/\!\!/N$. Then

$$\mathcal{A} = \mathrm{Maps}(C, \mathbb{A}^r/\mathbb{G}_m^r) = (\mathrm{Sym}\, C)^r = \coprod_{(n_i)\in\mathbb{N}^r} C^{(n_1)} \times \cdots \times C^{(n_r)} =: \coprod \mathcal{A}^{n_1\check{\nu}_1 + \cdots + n_r\check{\nu}_r}$$

is the scheme of r divisors on C. Let the preimage of $\mathcal{A}^{\check{\lambda}}$ be $\mathcal{Y}^{\check{\lambda}}$.

[4]Zastava is Croatian for flag

Then $\mathcal{Y}^{\check\lambda}$ is a finite type *scheme*.

3.2. **Graded factorization.** Notice that the fiber over $\check\lambda_1 \cdot v_1 + \check\lambda_2 \cdot v_2 \in \mathcal{A}^{\check\lambda_1+\check\lambda_2}$ where v_1, v_2 are distinct only depends on the independent fibers over $\check\lambda_1 \cdot v_1$ and $\check\lambda_2 \cdot v_2$. This is called a *graded factorization* property of (the collection of components of) $\mathcal{Y}$.

Aside: in the situation above the $\mathcal{Y}^{\check\lambda}$ are indeed irreducible components, but we could only prove this in a *very* roundabout way.

3.3. **Upshot: central fibers.** The graded factorization property essentially says the fiber of π over $\check\lambda \cdot v$ at a single point $v \in C(k)$ is the most important. This fiber is isomorphic to

$$\mathbb{Y}^{\check\lambda} := \mathrm{Gr}^{\check\lambda}_{B,v} \underset{\mathbf{X_F}/\mathbf{B_O}}{\times} \mathbf{X_O}/\mathbf{B_O},$$

where $\mathbf{B_F} \to \mathbf{X_F}$ is the action on x_0. This fiber doesn't depend on v. Observe that

$$tr(\mathrm{Fr}, \pi_! \mathrm{IC}_{\mathcal{Y}}|^*_{\check\lambda\cdot v}) = tr(\mathrm{Fr}, H^*_c(\mathbb{Y}^{\check\lambda}, \mathrm{IC}_{\mathcal{Y}})) = \int_{N(F)} \Phi_0(x_0 n t^{\check\lambda}) = \pi_!\Phi_0(t^{\check\lambda})$$

is the Radon transform we wanted to calculate back in (2.1).

Example 3. Let $X = \mathbb{G}_m\backslash\mathrm{GL}_2$ where $\mathbb{G}_m = \left(\begin{smallmatrix} * & \\ & 1\end{smallmatrix}\right)$. Then $\mathcal{Y} = \mathrm{Maps}_{\mathrm{gen}}(C, X/B) = \mathrm{Maps}_{\mathrm{gen}}(C, \mathbb{G}_m\backslash\mathbb{P}^1)$ parametrizes

$$\mathcal{A}, \mathcal{L} \in \mathrm{Pic}, \mathcal{L} \xrightarrow{(x,y)} \mathcal{A} \oplus \mathcal{O}.$$

Generically landing in X° means x, y do not simultaneously vanish after taking fiber at any point. What this amounts to is two divisors with disjoint support:

$$\mathcal{Y} = \mathrm{Sym}\, C \overset{\circ}{\times} \mathrm{Sym}\, C$$

Meanwhile $X/\!\!/N = \mathbb{A}^2$ with basis $\check\varepsilon_1 = (1,0), -\check\varepsilon_2 = (0,-1)$. So

$$\pi : \mathcal{Y} = \mathrm{Sym}\, C \overset{\circ}{\times} \mathrm{Sym}\, C \to \mathrm{Sym}\, C \times \mathrm{Sym}\, C = \mathcal{A}$$

is an open embedding. The preimage of $(n_1\check\varepsilon_1 - n_2\check\varepsilon_2)\cdot v$ is empty if n_1, n_2 are both nonzero, and a point otherwise. So we see

$$\pi_!\Phi_0 = e^0 + \sum_{n\geq 1}(q^{-n/2}e^{n\check\varepsilon_1} + q^{-n/2}e^{-n\check\varepsilon_2}) = \frac{1 - q^{-1}e^{\check\alpha}}{(1-q^{-1/2}e^{\check\varepsilon_1})(1-q^{-1/2}e^{-\check\varepsilon_2})}$$

since $\check\alpha = \check\varepsilon_1 - \check\varepsilon_2$. Note that $|\widehat{\pi_!\Phi_0}(\chi)|^2 = \frac{L(\chi, \mathrm{std}\oplus\mathrm{std}^*, 1/2)}{L(\chi, \check{\mathfrak{g}}/\check{\mathfrak{t}}, 1)}$.

As we see above, π is not proper, but we can compactify it to:

$$\overline{\mathcal{Y}} = \mathrm{Maps}(C, X \times \overline{G/N}/(G^{\mathrm{diag}} \times T))$$

and we still have $\bar\pi : \overline{\mathcal{Y}} \to \mathcal{A}$. Let $\overline{\mathcal{Y}}^{\check\lambda}$ be preimage of $\mathcal{A}^{\check\lambda}$. And $\overline{\mathcal{Y}}$ still has the graded factorization property.

Theorem 7 (Sakellaridis–W)**.** *Under our assumptions on X, the map $\bar\pi : \overline{\mathcal{Y}} \to \mathcal{A}$ is stratified semi-small.*

We emphasize that this is extremely special to the $\check G_X = \check G$ case! The statement is definitely false for example when $X = \overline{N^-\backslash G}$.

Toy situation: if $\overline{\mathcal{Y}}$ were smooth, then semi-smallness for $\bar\pi$ amounts to (because of factorization):

$$\dim \overline{\mathbb{Y}}^{\check\lambda} \leq \mathrm{crit}(\check\lambda) \tag{3.1}$$

The general situation is more complicated because of strata, but believe that we have some formula for $\operatorname{crit}(\check{\lambda})$.

3.3.1. A fact that is presumably known to experts but not often stated is that in the above situation where you have a semi-small map, the decomposition theorem together with the graded factorization property immediately tell you that

$$tr(\mathrm{Fr}, (\bar{\pi}_! \mathrm{IC}_{\overline{\mathcal{Y}}})|^*_{?\cdot v}) = \frac{1}{\prod_{\check{\lambda} \in \mathfrak{B}^+} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})}$$

has the desired Euler product format. Here $\mathfrak{B}^+$ corresponds to the *relevant strata* supported at a single point. More specifically, $\mathfrak{B}^+ =$ the irreducible components of $\overline{\mathbb{Y}}^{\check{\lambda}}$ of $\dim = \operatorname{crit}(\check{\lambda})$ as $\check{\lambda}$ varies. (This is an oversimplification but it's almost true.)

3.4. **Crystals.** To reconnect with Conjecture 2, define V_X^+ to be the $\check{T}$-representation with basis in bijection with $\mathfrak{B}^+$. The crux of Conjecture 2 is getting half of a $\check{G}$-representation.

Since we know this is what we want, formally set $\mathfrak{B} = \mathfrak{B}^+ \sqcup (\mathfrak{B}^+)^*$, where $(\mathfrak{B}^+)^*$ is defined to be in bijection with $\mathfrak{B}^+$ but the weights are replaced by their negatives. In this way, $(\mathfrak{B}^+)^*$ naturally corresponds to a basis of $(V_X^+)^*$.

Theorem 8 (Sakellaridis–W)**.** *$\mathfrak{B}$ has the structure of a (Kashiwara) crystal, i.e., a graph with weighted vertices and edges corresponding to lowering operators $\tilde{f}_\alpha$.*

We use this abstract combinatorial notion of crystal as a bridge to hopefully getting a crystal basis. A crystal basis is the (Lusztig) canonical basis[5] at $q = 0$ of an integrable $U_q(\check{\mathfrak{g}})$-module in category $\mathcal{O}$. So the crystal basis is a way for us to access a $\check{G}$-representation.

$$\text{f.d. } \check{G}\text{-representation} \rightsquigarrow \text{crystal basis} \in \{\text{crystals}\}$$

Conjecture 3. *$\mathfrak{B}$ is the crystal basis for a finite dimensional $\check{G}$-representation V_X.*

Conjecture 3 implies Conjecture 2 (by construction, $\mathfrak{B}$ corresponds to a basis of V'_X).

Conjecture 2 resembles geometric constructions of crystal bases by Lusztig, Braverman–Gaitsgory [BG01], and Kamnitzer. But in their situations they are concerned with constructing crystal bases for *all* representations, whereas here we arrive at a very specific one. I am interested in possible connections in this theory.

3.5. **Further details.** We can identify $(\mathrm{Gr}_B^{\check{\lambda}})_{red} = \mathbf{N_F} t^{\check{\lambda}} \mathbf{G_O}/\mathbf{G_O} =: S^{\check{\lambda}} \subset \mathrm{Gr}_G$, i.e., a semi-infinite orbit. Let $\overline{S}^{\check{\lambda}}$ denote its closure in Gr_G. Then the fiber of $\overline{\mathcal{Y}} \to \mathcal{A}$ over $\check{\lambda} \cdot v$ is

$$\overline{S}^{\check{\lambda}} \underset{\mathbf{X}_F/\mathbf{G_O}}{\times} \mathbf{X_O}/\mathbf{G_O}$$

Proposition 2 ([MV])**.** *The boundary $\overline{S}^{\check{\lambda}} = \bigcup_{\check{\mu} \leq \check{\lambda}} S^{\check{\mu}}$ is given by a hyperplane section in Gr_G.*

The lowering operator we define on $\mathfrak{B}$ is roughly given by

$$\overline{\mathbb{Y}}^{\check{\lambda}} \rightsquigarrow \overline{\mathbb{Y}}^{\check{\lambda}} \cap S^{\check{\lambda} - \check{\alpha}} \subset \mathbb{Y}^{\check{\lambda} - \check{\alpha}}.$$

This does not quite uniquely specify how to lower an irreducible component to another irreducible component, but a reduction to considering affine embeddings of $\mathbb{G}_m \backslash \mathrm{GL}_2 \times (\text{torus})$ gives us enough information to pick out the correct irreducible component in $\mathbb{Y}^{\check{\lambda} - \check{\alpha}}$.

[5] Canonical bases were first discovered by Lusztig '90 in types A, D, E, and subsequently by Kashiwara using different methods. The crystal basis at $q = 0$ in types A, B, C, D was discovered independently by Kashiwara at around the same time in '90.

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