Local L-values and geometric harmonic analysis on spherical varieties

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Outline

1. Integral representations of L-functions
2. What is a spherical variety?
3. Function-theoretic results
4. Geometry

G connected split reductive group $/\mathbb{F}_q$

Integral representations of L-functions

- C smooth, projective, geometrically connected curve over $\mathbb{F}_q$
- $\Bbbk = \mathbb{F}_q(C)$ global function field
- $[G] = G(\Bbbk)\backslash G(\mathbb{A})$

Automorphic period integral

For a "nice" reductive subgroup $H \subset G$, the period integral

$$\mathcal{P}_H(f) := \int_{[H]} f(h)dh$$

for f a cusp form on $[G]$ is related to a special value of an L-function.

In these cases, $X = H\backslash G$ is a homogeneous affine spherical variety.

Theorem (Luna, Richardson)

$H\backslash G$ is affine if and only if H is reductive.

By formal manipulation

$$\int_{[H]} f(h)dh = \int_{[G]} f(g) \cdot \sum_{\gamma \in (H \backslash G)(\Bbbk)} \mathbf{1}_{X(\mathbb{O})}(\gamma g) dg$$

where

Definition

$$\Sigma\Phi(g) := \sum_{\gamma \in H\backslash G(\Bbbk)} \Phi(\gamma g)$$

is the *X-Poincaré series (alias X-Theta series)* on $[G]$

$$X(\mathbb{A}) \leftarrow (H\backslash G)(\Bbbk) \overset{G(\Bbbk)}{\times} G(\mathbb{A}) \rightarrow G(\Bbbk)\backslash G(\mathbb{A}) = [G]$$

General expectation (Sakellaridis)

- Start with $X^\bullet = H\backslash G$ "nice" (H not necessarily reductive)
- Choose an affine embedding $X^\bullet \hookrightarrow X$ (e.g., $X = \overline{X^\bullet}^{\mathrm{aff}}$)
- Let $\Phi_0 = IC_{X(\mathbb{O})}$ denote the "IC function" of $X(\mathbb{O})$
- Define the X-Poincaré series

$$\Sigma\Phi_0(g) = \sum_{\gamma \in X^\bullet(\Bbbk)} \Phi_0(\gamma g)$$

- Define the "X-period" by

$$\mathcal{P}_X(f) = \int_{[G]} f \cdot \Sigma\Phi_0, \qquad f \text{ cusp form on } [G]$$

Conjecture (Sakellaridis, 2009)

If f is unramified, then $|\mathcal{P}_X(f)|^2$ is "equal" to special value of L-function.

Example (Rankin–Selberg convolution)

For π_1, π_2 cuspidal $\mathrm{GL}_2(\mathbb{A})$-representations,

$$L(\tfrac{1}{2}+s, \pi_1 \times \pi_2, \mathrm{std} \otimes \mathrm{std}) = \int_{Z(\mathbb{A})\backslash[\mathrm{GL}_2]} f_1(g) f_2(g) E^*(g, \tfrac{1}{2}+s) dg$$

for unramified $f_1 \in \pi_1, f_2 \in \pi_2$, Whittaker normalized.

- Think of the normalized Eisenstein series $E^*(g,s) = \zeta(2s)E(g,s)$ as a distribution on $[\mathrm{GL}_2 \times \mathrm{GL}_2]$ via diagonal embedding.
- RHS is obtained by Mellin transform from

$$\mathcal{P}_X(f_1 \times f_2) = \int_{[G]} (f_1 \times f_2) \cdot \Sigma(\mathbf{1}_{X(\mathbb{O})})$$

  - $G = \mathrm{GL}_2 \times \mathrm{GL}_2 \circlearrowright X = \mathbb{A}^2 \times \mathrm{GL}_2$
  - open G-orbit $X^\bullet = (\mathbb{A}^2 - 0) \times \mathrm{GL}_2 = H\backslash G$
  - $H = \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ mirabolic subgroup, diagonally embedded

In all the previous examples, X was smooth.

Example (Sakellaridis)

$G = \mathrm{GL}_2^{\times n} \times \mathbb{G}_m$, $H =$

$$\left\{ \begin{pmatrix} a & x_1 \\ & 1 \end{pmatrix} \times \begin{pmatrix} a & x_2 \\ & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} a & x_n \\ & 1 \end{pmatrix} \times a \,\middle|\, x_1 + \cdots + x_n = 0 \right\}$$

Let $X = \overline{H \backslash G}^{\mathrm{aff}}$ (usually singular).

- $n = 2$: Rankin–Selberg
- $n = 3$: $\mathcal{P}_X$ is equivalent to the construction of Garrett

This is a case where the integral $\mathcal{P}_X$ "unfolds" and our local results imply:

Theorem (Sakellaridis-W)

Over a global function field, the Mellin transform of $\mathcal{P}_X|_\pi$ gives an integral representation of $L(s, \pi, \mathrm{std}^{\otimes n} \otimes \mathrm{std}_1)$ for $\mathrm{Re}(s) \gg 0$ on cuspidal representations π under Whittaker normalization.

What is a spherical variety?

$k = \overline{\mathbb{F}}_q$

Definition

A G-variety $X_{/\mathbb{F}_q}$ is called spherical if X_k is normal and has an open dense orbit of $B_k \subset G_k$ after base change to k

Think of this as a finiteness condition (good combinatorics)

Examples:

- Toric varieties $G = T$
- Symmetric spaces $K \backslash G$
  - Group $X = G' \circlearrowleft G' \times G' = G$
- Reductive monoid $X \supsetneq X^{\bullet} = G' \circlearrowleft G' \times G'$

Why are they relevant?

Conjecture (Sakellaridis–Venkatesh)

For any affine spherical G-variety X (*), and a cuspidal $G(\mathbb{A})$-representation $\pi \hookrightarrow \mathcal{A}_0(G)$,

1. $\mathcal{P}_X|_\pi \neq 0$ *implies that* π *lifts to* $\sigma \hookrightarrow \mathcal{A}_0(G_X)$ *by functoriality along a map* $\check{G}_X(\mathbb{C}) \to \check{G}(\mathbb{C})$*,*
2. *there should exist a* $\check{G}_X$*-representation*

$$\rho_X : \check{G}_X(\mathbb{C}) \to \mathrm{GL}(V_X)$$

such that $|\mathcal{P}_X|_\pi^2 = (*)\frac{L(s_0,\sigma,\rho_X)}{L(1,\sigma,\mathrm{Ad})}$ *for a special value* s_0*.*

Some history on $\check{G}_X$

Goal: a map $\check{G}_X \to \check{G}$ with finite kernel

- $\check{T}_X$ is easy to define
- Little Weyl group W_X and spherical root system
  - Symmetric variety: Cartan '27
  - Spherical variety: Brion '90, Knop '90, '93, '94
- Gaitsgory–Nadler '06: define subgroup $\check{G}_X^{GN} \subset \check{G}$ using Tannakian formalism
- Sakellaridis–Venkatesh '12: normalized root system, define $\check{G}_X \to \check{G}$ combinatorially with image $\check{G}_X^{GN}$ under assumptions about GN
- Knop–Schalke '17: define $\check{G}_X \to \check{G}$ combinatorially unconditionally

	$X \circlearrowleft G$	$\check{G}_X$	V_X
Usual Langlands	$G' \circlearrowleft G' \times G'$	$\check{G}'$	$\check{\mathfrak{g}}'$
Whittaker normalization	$(N,\psi)\backslash G$	$\check{G}$	0
Hecke	$\mathbb{G}_m\backslash \mathrm{PGL}_2$	$\check{G} = \mathrm{SL}_2$	$T^*\mathrm{std}$
Rankin–Selberg, Jacquet–Piatetski-Shapiro–Shalika	$\overline{H\backslash \mathrm{GL}_n \times \mathrm{GL}_n} = \mathrm{GL}_n \times \mathbb{A}^n$	$\check{G}$	$T^*(\mathrm{std}\otimes\mathrm{std})$
Gan–Gross–Prasad	$\mathrm{SO}_{2n}\backslash \mathrm{SO}_{2n+1}\times \mathrm{SO}_{2n}$	$\check{G} = \mathrm{SO}_{2n} \times \mathrm{Sp}_{2n}$	$\mathrm{std} \otimes \mathrm{std}$

$\check{G}_X = \check{G}$

For this talk, assume $\check{G}_X = \check{G}$ (and X has no type N roots). ['N' is for normalizer]

Equivalent to:

(Base change to k)

- X has open B-orbit $X^\circ \cong B$
- $X^\circ P_\alpha / \mathcal{R}(P_\alpha) \cong \mathbb{G}_m \backslash \mathrm{PGL}_2$ for every simple α, $P_\alpha \supset B$

Sakellaridis–Venkatesh á la Bernstein

Sakellaridis–Venkatesh: give generalized Ichino–Ikeda conjecture relating $|\mathcal{P}_X|^2$ to local harmonic analysis:

$$|\mathcal{P}_X(f)|^2 \overset{\text{I-I conjecture}}{=} \prod_v (\text{local computation})$$

$F = \mathbb{F}_q((t))$, $O = \mathbb{F}_q[[t]]$

- spherical functions (unramified Hecke eigenfunction) on $X^\bullet(F)$
- unramified Plancherel measure on $X^\bullet(F)$

Fix $x_0 \in X^\circ(\mathbb{F}_q)$ in open B-orbit. For $\Phi \in C_c^\infty(X(F))^{G(O)}$, define the X-Radon transform

$$\pi_! \Phi(g) := \int_{N(F)} \Phi(x_0 n g) d n, \quad g \in G(F)$$

$\pi_! \Phi$ is a function on $N(F) \backslash G(F) / G(O) = T(F)/T(O) = \check{\Lambda}$.

Conjecture 1 (Sakellaridis–Venkatesh)

Assume $\check{G}_X = \check{G}$ *and* X *has no type* N *roots.*
There exists a symplectic $V_X \in \mathrm{Rep}(\check{G})$ *with a* $\check{T}$ *polarization* $V_X = V_X^+ \oplus (V_X^+)^*$ *such that*

$$\pi_! IC_{X(O)} = \frac{\prod_{\check{\alpha} \in \check{\Phi}_G^+} (1 - q^{-1} e^{\check{\alpha}})}{\prod_{\check{\lambda} \in \mathrm{wt}(V_X^+)} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})} \in \mathrm{Fn}(\check{\Lambda})$$

where $e^{\check{\lambda}}$ *is the indicator function of* $\check{\lambda}$*,* $e^{\check{\lambda}} e^{\check{\mu}} = e^{\check{\lambda} + \check{\mu}}$

Mellin transform of right hand side gives

$$\chi \in \check{T}(\mathbb{C}) \mapsto \frac{L(\frac{1}{2}, \chi, V_X^+)}{L(1, \chi, \check{\mathfrak{n}})}, \text{ this is "half" of } \frac{L(\frac{1}{2}, \chi, V_X)}{L(1, \chi, \check{\mathfrak{g}}/\check{\mathfrak{t}})}$$

Previous work

Conjecture 1 (possibly with $\check{G}_X \neq \check{G}$) was proved in the following cases:

- Sakellaridis ('08, '13):
  - $X = H\backslash G$ and H is reductive (iff $H\backslash G$ is affine), no assumption on $\check{G}_X$
  - doesn't consider $X \supsetneq H\backslash G$
- Braverman–Finkelberg–Gaitsgory–Mirković [BFGM] '02:
  - $X = \overline{N^-\backslash G}$, $\check{G}_X = \check{T}$, $V_X = \check{\mathfrak{n}}$
- Bouthier–Ngô–Sakellaridis [BNS] '16:
  - X toric variety, $G = T$, $\check{G}_X = \check{T}$, weights of V_X correspond to lattice generators of a cone
  - $X \supset G'$ is L-monoid, $G = G' \times G'$, $\check{G}_X = \check{G}'$, $V_X = \check{\mathfrak{g}}' \oplus T^*V^{\check{\lambda}}$

Theorem (Sakellaridis–W)

Assume X affine spherical, $\check{G}_X = \check{G}$ and X has no type N roots. Then

$$\pi_! IC_{X(O)} = \frac{\prod_{\check{\alpha} \in \check{\Phi}_G^+} (1 - q^{-1} e^{\check{\alpha}})}{\prod_{\check{\lambda} \in \mathrm{wt}(V_X^+)} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})}$$

for some $V_X^+ \in \mathrm{Rep}(\check{T})$ such that:

1. *(Functional equation) $V_X := V_X^+ \oplus (V_X^+)^*$ has action of $(\mathrm{SL}_2)_\alpha$ for every simple root α*
   - *We do not check Serre relations*
2. *Assuming V_X satisfies Serre relations (so it is a $\check{G}$-representation), we determine its highest weights with multiplicities (in terms of X)*

- (2) gives recipe for conjectural (ρ_X, V_X) in terms of only data from X
- If V_X is minuscule, then Serre relations hold

Proposition

If $X = H\backslash G$ with H reductive, then V_X is minuscule.

Enter geometry

- Base change to $k = \overline{\mathbb{F}}_q$ (or $k = \mathbb{C}$)
- $\mathbf{X_O}(k) = X(k[\![t]\!])$
- Problem: $\mathbf{X_O}$ is an infinite type scheme
- Bouthier–Ngô–Sakellaridis: IC function still makes sense by Grinberg–Kazhdan theorem

Zastava space

Drinfeld's proof of Grinberg–Kazhdan theorem gives an explicit model for $\mathbf{X_O}$:

Definition

Let C be a smooth curve over k. Define

$$\mathcal{Y} = \{y : C \to X/B \text{ generically landing in } X^\circ/B = \text{pt}\} \subset \prod_{v \in |C|}' X(O_v)/B(O_v)$$

Following Finkelberg–Mirković, we call this the **Zastava space** of X.

Fact: $\mathcal{Y}$ is an infinite disjoint union of finite type schemes.

$$\mathcal{Y} \xrightarrow{\pi} \mathcal{A} \subset \{\check{\Lambda}\text{-valued divisors on } C\}$$

Definition

Define the **central fiber** $\mathbb{Y}^{\check{\lambda}} = \pi^{-1}(\check{\lambda} \cdot v)$ for a single point $v \in C(k)$.

$$\begin{array}{ccc} \mathcal{Y} & \longleftarrow & \mathbb{Y}^{\check{\lambda}} \\ \big\downarrow & & \big\downarrow \\ \mathcal{A} & \longleftarrow & \check{\lambda} \cdot v \end{array}$$

Integrals $\rightsquigarrow$ cohomology

$$\pi_! IC_{\mathbf{X_0}}(t^{\check{\lambda}}) = tr(\mathrm{Fr}, (\pi_! \mathrm{IC}_{\mathcal{Y}})|^*_{\check{\lambda}\cdot v})$$

Can compactify π to a proper map $\bar{\pi} : \overline{\mathcal{Y}} \to \mathcal{A}$.

Graded factorization property

The fiber $\bar{\pi}^{-1}(\check{\lambda}_1 v_1 + \check{\lambda}_2 v_2)$ for distinct v_1, v_2 is equal to $\overline{\mathbb{Y}}^{\check{\lambda}_1} \times \overline{\mathbb{Y}}^{\check{\lambda}_2}$.

Decomposition theorem + factorization property imply

Euler product

$$tr(\mathrm{Fr}, (\bar{\pi}_! \mathrm{IC}_{\overline{\mathcal{Y}}})|^*_{?\cdot v}) = \frac{1}{\prod_{\check{\lambda} \in \mathfrak{B}^+} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})}$$

- $q^{-\frac{1}{2}} \leftrightarrow \bar{\pi}$ is stratified **semi-small**
- $\mathfrak{B}^+ =$ irred. components of $\overline{\mathbb{Y}}^{\check{\lambda}}$ of dim $= \mathrm{crit}(\check{\lambda})$ as $\check{\lambda}$ varies

Define V_X^+ to have basis $\mathfrak{B}^+$.

The $(\mathrm{SL}_2)_\alpha$-action on $V_X^+ \oplus (V_X^+)^*$ is defined by a reduction to the Hecke case $\mathbb{G}_m \backslash \mathrm{GL}_2$.