# UNIVERSAL LOCAL ACYCLICITY

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In this note, $k$ can be any perfect field. We recall the definition of *universal local acyclicity* as in [Del77]. Let $S$ be a scheme and $s$ a geometric point of $S$. We denote by $S_{(s)}$ the strict Henselisation of $S$ at $s$. We will formally write $t \to s$ if $t$ is a geometric point of $S_{(s)}$.

**Definition 1.** Let $f : Y \to S$ be a morphism of schemes of finite type over $k$. An object $\mathcal{F} \in \mathrm{D}_c^b(Y)$ is called *locally acyclic* with respect to $f$ if for every geometric point $y$ of $Y$ and every specialisation $t \to f(y)$, the natural map $R\Gamma(Y_{(y)}, \mathcal{F}) \to R\Gamma(Y_{(y)} \times_{S_{(f(y))}} t, \mathcal{F})$ is an isomorphism.

It is called universally locally acylic (ULA) if it is locally acyclic after arbitrary base change $S' \to S$.

We refer the reader to [Del77], [Zhu17, §A.2] for a review of the ULA property. In particular, the property is local in the smooth topology on the source and target (cf. [Zhu17, Theorem A.2.5]), meaning: Let $f : Y \to S$ be a morphism of finite type $k$-schemes and $\mathcal{F} \in \mathrm{D}_c^b(Y)$.

1. If $g : Y' \to Y$ is a smooth (resp. smooth and surjective) map, then $g^*(\mathcal{F}) \in \mathrm{D}_c^b(Y')$ is ULA with respect to $f \circ g : Y' \to S$ if (resp. if and only if) $\mathcal{F}$ is ULA with respect to $f : Y \to S$.
2. If $g : S \to S'$ is a smooth map and $\mathcal{F}$ is ULA with respect to $f : Y \to S$, then $\mathcal{F}$ is ULA with respect to $g \circ f : Y \to S'$.

Therefore, it makes sense to extend the definition of ULA to morphisms between algebraic stacks of finite type over $k$. We continue to work with schemes in this appendix, but one can easily generalize the statements to stacks.

In [BG02, §5.1, Theorem B.2], the authors introduced an equivalent definition (Definition 2 below) of locally acyclic complexes when the base $S$ is smooth. In this appendix we will prove some properties of ULA complexes when the base $S$ is possibly *not* smooth, following the arguments in the proof of [BG02, Theorem B.2], with the goal of proving Lemma 7.

First, we have the following reformulation of Definition 1 by [BG02]:

Let $j : S_0 \hookrightarrow S$ be a smooth locally closed subvariety and let $\mathcal{L}$ be a lisse sheaf on $S_0$. Consider the Cartesian diagram

$$\begin{array}{ccc} Y_0 & \xrightarrow{f_0} & S_0 \\ \big\downarrow{\scriptstyle j'} & & \big\downarrow{\scriptstyle j} \\ Y & \xrightarrow{f} & S \end{array}$$

By the $(j'^*, j'_*)$-adjunction we have a natural map

$$\mathcal{F} \otimes f^*(j_*(\mathcal{L})) \to j'_*(j'^*(\mathcal{F}) \otimes f_0^*(\mathcal{L})). \tag{1.1}$$

One observes that $\mathcal{F} \in \mathrm{D}_c^b(Y)$ is locally acyclic with respect to $f$ if and only if for all $S_0$ and $\mathcal{L}$ as above, the map (1.1) is an isomorphism.

Now let $\mathcal{G} \in \mathrm{D}_c^b(S)$ be arbitrary. We have a natural map

$$f^*(\mathcal{G}) \otimes f^!(\overline{\mathbb{Q}}_\ell) \to f^!(\mathcal{G}), \tag{1.2}$$

which comes by adjunction from the map

$$f_!(f^*(\mathcal{G}) \otimes f^!(\overline{\mathbb{Q}}_\ell)) \cong \mathcal{G} \otimes f_! f^!(\overline{\mathbb{Q}}_\ell) \to \mathcal{G} \otimes \overline{\mathbb{Q}}_\ell.$$

The map (1.2) induces a natural map

$$\underline{\mathrm{Hom}}(\mathcal{F}, f^!(\overline{\mathbb{Q}}_\ell)) \otimes f^*\mathcal{G} \to \underline{\mathrm{Hom}}(\mathcal{F}, f^!\overline{\mathbb{Q}}_\ell \otimes f^*\mathcal{G}) \xrightarrow{(1.2)} \underline{\mathrm{Hom}}(\mathcal{F}, f^!\mathcal{G}) \tag{1.3}$$

where $\underline{\mathrm{Hom}}$ is the (derived) internal Hom.

Let $\mathbb{D}$ denote the duality functor. By Grothendieck's six functor formalism, there is a functorial isomorphism

$$\mathbb{D}(\mathcal{F}_1 \otimes \mathbb{D}\mathcal{F}_2) \cong \underline{\mathrm{Hom}}(\mathcal{F}_1, \mathcal{F}_2), \qquad \mathcal{F}_1, \mathcal{F}_2 \in \mathrm{D}^b_c(Y) \tag{1.4}$$

([SGA77, Exposé I, Proposition 1.11(c)]). Therefore, we have $\underline{\mathrm{Hom}}(\mathcal{F}, f^!\mathcal{G}) \cong \mathbb{D}(\mathcal{F} \otimes f^*\mathbb{D}\mathcal{G})$.

Let $\mathcal{F}_1 \otimes^! \mathcal{F}_2 := \mathbb{D}(\mathbb{D}\mathcal{F}_1 \otimes \mathbb{D}\mathcal{F}_2)$ denote the conjugate of $\otimes$ by $\mathbb{D}$.

If $S$ is smooth, then $\underline{\mathrm{Hom}}(\mathcal{F}, f^!(\overline{\mathbb{Q}}_\ell)) = \mathbb{D}(\mathcal{F})(-d)[-2d]$ where $d : \pi_0(S) \to \mathbb{Z}$ is the dimension. Then using (1.4), we deduce that the Verdier dual of (1.3) is a map

$$\mathcal{F} \otimes f^*(\mathbb{D}\mathcal{G}) \to \mathcal{F} \overset{!}{\otimes} f^!(\mathbb{D}\mathcal{G})(d)[2d]. \tag{1.5}$$

Then [BG02, Theorem B.2] showed that Definition 1 is equivalent to the following:

**Theorem 2.** *Let $f : Y \to S$ be a morphism of schemes of finite type over $k$ where $S$ is smooth. An object $\mathcal{F} \in \mathrm{D}^b_c(Y)$ is locally acyclic with respect to $f$ if and only if (1.5) is an isomorphism for every $\mathcal{G} \in \mathrm{D}^b_c(S)$.*

We now use Theorem 2 to deduce several properties of ULA complexes with respect to morphisms $f : Y \to S$ where the base $S$ is not necessarily smooth. These properties are presumably known to experts[1] but we provide proofs as they do not seem to appear in the literature.

**Proposition 3.** *If $\mathcal{F}$ is ULA with respect to $f : Y \to S$, then for every $\mathcal{G} \in \mathrm{D}^b_c(S)$, the natural map (1.3) is an isomorphism*

$$\underline{\mathrm{Hom}}(\mathcal{F}, f^!\overline{\mathbb{Q}}_\ell) \otimes f^*\mathcal{G} \cong \underline{\mathrm{Hom}}(\mathcal{F}, f^!\mathcal{G}).$$

*Proof.* We may assume $S$ is reduced and hence generically smooth. Let $j : S_0 \hookrightarrow S$ be a smooth open dense subvariety of $S$ such that $j^*\mathcal{G}$ has lisse cohomology sheaves. Let $i : S_1 \hookrightarrow S$ denote the complementary closed embedding. Let $j' : Y_0 \hookrightarrow Y$ and $i' : Y_1 \hookrightarrow Y$ denote the corresponding embeddings after base change to $Y$ and $f_i : Y_i \to S_i$ the projection. The $j_! j^! \to 1 \to i_* i^*$ distinguished triangle applied to $(\overline{\mathbb{Q}}_\ell)_S$ and $\mathcal{G}$ gives a map of distinguished triangles

(3.1)

$$\begin{array}{ccccc} \underline{\mathrm{Hom}}(\mathcal{F}, f^! j_!(\overline{\mathbb{Q}}_\ell)_{S_0}) \otimes f^*\mathcal{G} & \longrightarrow & \underline{\mathrm{Hom}}(\mathcal{F}, f^!\overline{\mathbb{Q}}_\ell) \otimes f^*\mathcal{G} & \longrightarrow & \underline{\mathrm{Hom}}(\mathcal{F}, i'_* f_1^!(\overline{\mathbb{Q}}_\ell)_{S_1}) \otimes f^*\mathcal{G} \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(\mathcal{F}, f^! j_! j^*\mathcal{G}) & \longrightarrow & \underline{\mathrm{Hom}}(\mathcal{F}, f^!\mathcal{G}) & \longrightarrow & \underline{\mathrm{Hom}}(\mathcal{F}, i'_* f_1^! i^*\mathcal{G}). \end{array}$$

We will show that the left and right vertical arrows are isomorphisms, which implies the middle arrow is also an isomorphism.

[1] We learned of the statements from [KHW17, Theorem 4.6.3], which lives in the more nuanced setting of $p$-adic geometry.

By (1.4) we have $\underline{\mathrm{Hom}}(\mathcal{F}, f^! j_!(\overline{\mathbb{Q}}_\ell)_{S_0}) \cong \mathbb{D}(\mathcal{F} \otimes f^* j_*(\overline{\mathbb{Q}}_\ell)_{S_0}(d_0)[2d_0])$, where we have used that $S_0$ is smooth of dimension $d_0 : \pi_0(S_0) \to \mathbb{Z}$. Therefore, the isomorphism of (1.1) gives

$$\underline{\mathrm{Hom}}(\mathcal{F}, f^! j_!(\overline{\mathbb{Q}}_\ell)_{S_0}) \otimes f^* \mathcal{G} \cong j'_! \Big(\mathbb{D}(j'^* \mathcal{F}) \otimes f_0^*(j^* \mathcal{G})\Big)(-d_0)[-2d_0]. \tag{3.2}$$

Since $\mathcal{F}$ is ULA, $\mathbb{D}(j'^*\mathcal{F})$ is ULA with respect to $f_0 : Y_0 \to S_0$ where $S_0$ is smooth. Thus, using Theorem 2, we have a canonical isomorphism

$$\mathbb{D}(j'^* \mathcal{F}) \otimes f_0^*(j^* \mathcal{G})(-d_0)[-2d_0] \cong \mathbb{D}(j'^*(\mathcal{F})) \overset{!}{\otimes} f_0^!(j^* \mathcal{G}) \cong j'^* \underline{\mathrm{Hom}}(\mathcal{F}, f^! \mathcal{G}),$$

where we are using (1.4) in the second isomorphism. To summarize, we have a canonical isomorphism $\underline{\mathrm{Hom}}(\mathcal{F}, f^! j_!(\overline{\mathbb{Q}}_\ell)_{S_0}) \otimes f^* \mathcal{G} \cong j'_! j'^* \underline{\mathrm{Hom}}(\mathcal{F}, f^! \mathcal{G})$. On the other hand, since $j^* \mathcal{G}$ is lisse, the same argument as above using the isomorphism (1.3) gives an isomorphism

$$\underline{\mathrm{Hom}}(\mathcal{F}, f^! j_! j^* \mathcal{G}) \cong j'_! \underline{\mathrm{Hom}}(j'^* \mathcal{F}, f_0^! j^* \mathcal{G}) \cong j'_! j'^* \underline{\mathrm{Hom}}(\mathcal{F}, f^! \mathcal{G}).$$

Thus, we have shown that the left vertical arrow in (3.1) is an isomorphism.

For the right vertical arrow, we have a natural isomorphism

$$\underline{\mathrm{Hom}}(\mathcal{F}, i'_* f_1^! (\overline{\mathbb{Q}}_\ell)_{S_1}) \otimes f^* \mathcal{G} \cong i'_* \underline{\mathrm{Hom}}(i'^* \mathcal{F}, f_1^! (\overline{\mathbb{Q}}_\ell)_{S_1}) \otimes f_1^*(i^* \mathcal{G})$$

by adjunction and projection formula. Since $\mathcal{F}$ is ULA, after base change $i'^* \mathcal{F}$ is ULA with respect to $f_1$. Hence by noetherian induction on $\dim(S)$ we may assume that we have a canonical isomorphism

$$\underline{\mathrm{Hom}}(i'^* \mathcal{F}, f_1^! (\overline{\mathbb{Q}}_\ell)_{S_1}) \otimes f_1^*(i^* \mathcal{G}) \cong i'_* \underline{\mathrm{Hom}}(i'^* \mathcal{F}, f_1^!(i^* \mathcal{G})) \cong \underline{\mathrm{Hom}}(\mathcal{F}, f^! i_* i^* \mathcal{G}),$$

which is the isomorphism of the right vertical arrow in (3.1). □

Now consider a Cartesian diagram of schemes of finite type over $k$:

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & S' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & S \end{array} \tag{3.3}$$

For $\mathcal{F} \in \mathrm{D}_c^b(Y)$, there is a natural map

$$(g')^* \underline{\mathrm{Hom}}(\mathcal{F}, f^! \overline{\mathbb{Q}}_\ell) \to \underline{\mathrm{Hom}}((g')^* \mathcal{F}, (f')^! \overline{\mathbb{Q}}_\ell) \tag{3.4}$$

which comes by the $(g'^*, g'_*)$-adjunction from the map

$$\underline{\mathrm{Hom}}(\mathcal{F}, f^! \overline{\mathbb{Q}}_\ell) \to \underline{\mathrm{Hom}}(\mathcal{F}, f^! g_* g^* \overline{\mathbb{Q}}_\ell) \cong \underline{\mathrm{Hom}}(\mathcal{F}, g'_*(f')^! \overline{\mathbb{Q}}_\ell) \cong g'_* \underline{\mathrm{Hom}}((g')^* \mathcal{F}, (f')^! \overline{\mathbb{Q}}_\ell),$$

where we used proper base change in the second arrow.

**Proposition 4.** *In the setup above, if $\mathcal{F}$ is ULA with respect to $f$, then the natural map* (3.4) *is an isomorphism*

$$(g')^* \underline{\mathrm{Hom}}(\mathcal{F}, f^! \overline{\mathbb{Q}}_\ell) \cong \underline{\mathrm{Hom}}((g')^* \mathcal{F}, (f')^! \overline{\mathbb{Q}}_\ell).$$

*Proof.* The assertion is Zariski-local on $S'$, so we may assume that $S, S'$ are affine and $g$ factors as $S' \overset{i}{\hookrightarrow} \mathbb{A}^n \times S \xrightarrow{\mathrm{pr}_2} S$ where $i$ is a closed embedding. The isomorphism of (3.4) when $g$ is smooth follows from the isomorphism

$$(g')^! \underline{\mathrm{Hom}}(\mathcal{F}, f^! \overline{\mathbb{Q}}_\ell) \cong \underline{\mathrm{Hom}}((g')^* \mathcal{F}, (g')^! f^! \overline{\mathbb{Q}}_\ell),$$

which always holds in the six functor formalism. Therefore, after base change along $\mathrm{pr}_2 : \mathbb{A}^n \times S \to S$, we may assume that $g : S' \hookrightarrow S$ is a closed embedding and $S', S$ are reduced. The open complement $S - S'$ is nonempty (otherwise we are done) so there exists a smooth dense open subvariety $j : S_0 \hookrightarrow S$ inside $S - S'$. Let $i : S_1 \hookrightarrow S$ denote the complement of $S_0$, so

$g$ factors as $S' \overset{g_1}{\hookrightarrow} S_1 \overset{i}{\hookrightarrow} S$. Let $i', j', g'_1$ denote the preimages in $Y$ and $f_i : Y_i \to S_i$ the base change of $f$. The $j_! j^! \to 1 \to i_* i^*$ distinguished triangle induces a distinguished triangle

$$(g')^* \underline{\mathrm{Hom}}(\mathcal{F}, f^! j_! \overline{\mathbb{Q}}_\ell) \to (g')^* \underline{\mathrm{Hom}}(\mathcal{F}, f^! \overline{\mathbb{Q}}_\ell) \to (g'_1)^* \underline{\mathrm{Hom}}(i'^* \mathcal{F}, f_1^! \overline{\mathbb{Q}}_\ell)$$

Just as in the proof of Proposition 3, (3.2), we have a canonical isomorphism

$$\underline{\mathrm{Hom}}(\mathcal{F}, f^! j_! \overline{\mathbb{Q}}_\ell) \cong j'_! j'^! (\mathbb{D}\mathcal{F})(-d_0)[-2d_0].$$

Therefore, the leftmost term in the distinguished triangle vanishes, and hence the second arrow is an isomorphism. Now $i'^* \mathcal{F}$ is ULA with respect to $Y_1 \to S_1$, so by induction on the codimension of $S'$ in $S$, we conclude that

$$(g')^* \underline{\mathrm{Hom}}(\mathcal{F}, f^! \overline{\mathbb{Q}}_\ell) \cong (g'_1)^* \underline{\mathrm{Hom}}(i'^* \mathcal{F}, f_1^! \overline{\mathbb{Q}}_\ell) \cong \underline{\mathrm{Hom}}(g'^* \mathcal{F}, f'^! \overline{\mathbb{Q}}_\ell),$$

and this isomorphism coincides with (3.4). ☐

Again in the setting of the diagram (3.3), given $\mathcal{F} \in \mathrm{D}_c^b(Y), \mathcal{F}' \in \mathrm{D}_c^b(S')$ we let

$$\mathcal{F} \underset{S}{\boxtimes} \mathcal{F}' := (g')^* \mathcal{F} \otimes (f')^* \mathcal{F}'(-\tfrac{d}{2})[-d] \in \mathrm{D}_c^b(Y')$$

where $d : \pi_0(S) \to \mathbb{Z}$ denotes the dimension of each connected component of $S$.

**Corollary 5.** *In the diagram (3.3), assume that $S$ is rationally smooth, i.e., the dualizing complex of $S$ is isomorphic to $\overline{\mathbb{Q}}_\ell(d)[2d]$. Let $\mathcal{F} \in \mathrm{D}_c^b(Y), \mathcal{F}' \in \mathrm{D}_c^b(S')$ and assume that $\mathcal{F}$ is ULA with respect to $f : Y \to S$. Then there is a natural isomorphism*

$$\mathbb{D}\mathcal{F} \underset{S}{\boxtimes} \mathbb{D}\mathcal{F}' \cong \mathbb{D}(\mathcal{F} \underset{S}{\boxtimes} \mathcal{F}').$$

*Proof.* We have the sequence of isomorphisms

$$\begin{aligned}
\mathbb{D}\mathcal{F} \underset{S}{\boxtimes} \mathbb{D}\mathcal{F}' &= (g')^* (\mathbb{D}\mathcal{F}) \otimes (f')^* (\mathbb{D}\mathcal{F}')(-\tfrac{d}{2})[-d] \\
&= (g')^* \underline{\mathrm{Hom}}\Big(\mathcal{F}, f^! \overline{\mathbb{Q}}_\ell(\tfrac{d}{2})[d]\Big) \otimes f'^* (\mathbb{D}\mathcal{F}') \\
&\cong \underline{\mathrm{Hom}}((g')^* \mathcal{F}, (f')^! \overline{\mathbb{Q}}_\ell(\tfrac{d}{2})[d]) \otimes f'^* (\mathbb{D}\mathcal{F}') && \text{(Prop. 4)} \\
&\cong \underline{\mathrm{Hom}}((g')^* \mathcal{F}, (f')^! \mathbb{D}\mathcal{F}')(\tfrac{d}{2})[d] && \text{(Prop. 3)} \\
&\cong \mathbb{D}((g')^* \mathcal{F} \otimes (f')^* \mathcal{F}'(-\tfrac{d}{2})[-d]) && (1.4) \\
&= \mathbb{D}(\mathcal{F} \underset{S}{\boxtimes} \mathcal{F}')
\end{aligned}$$

where we are applying Proposition 3 to $(g')^* \mathcal{F}$, which is ULA with respect to $f' : Y' \to S'$. ☐

**Lemma 6.** *In the diagram (3.3), assume that $S$ is smooth. Let $\mathcal{F} \in \mathrm{D}_c^b(Y), \mathcal{F}' \in \mathrm{D}_c^b(S')$ and assume that $\mathcal{F}$ is ULA with respect to $f : Y \to S$. If $\mathcal{F} \in {}^p\mathrm{D}^{\leq 0}(Y)$ and $\mathcal{F}' \in {}^p\mathrm{D}^{\leq 0}(S')$, then*

$$\mathcal{F} \underset{S}{\boxtimes} \mathcal{F}' = (g')^* \mathcal{F} \otimes (f')^* \mathcal{F}' \in {}^p\mathrm{D}^{\leq 0}(Y'). \tag{6.1}$$

*Proof.* By taking open neighborhoods of $S$ and $S'$, we may assume that $g$ factors as $S' \hookrightarrow \mathbb{A}^n \times S \xrightarrow{\mathrm{pr}_2} S$ where $S' \hookrightarrow \mathbb{A}^n \times S$ is a closed embedding. Replacing $\mathcal{F}$ and $S$ with $\mathrm{pr}_2'^*(\mathcal{F})(\frac{n}{2})[n]$ and $\mathbb{A}^n \times S$, respectively, we reduce to the case when $g$ is a closed embedding.

By decomposing $\mathcal{F}'$ in the derived category with respect to a smooth stratification of $S'$, we may assume that $g : S' \hookrightarrow S$ is a smooth locally closed embedding and $\mathcal{F}'$ has lisse (usual) cohomology sheaves. If $\dim(S') < \dim(S)$, there exists an open $U \subset S$ and a smooth function $u : U \to \mathbb{A}^1$ such that $U \cap S' \subset u^{-1}(0)$. By induction on $\dim(S)$, we assume that (6.1) holds when $*$-restricted to $S - U$, so we can replace $S$ with $U$. Then we have $S' \hookrightarrow S_0 \overset{i}{\hookrightarrow} S$, where

$S_0 = u^{-1}(0)$ is smooth of dimension $\dim(S)-1$ and $i$ is a closed embedding. Now $\mathcal{F}$ is ULA with respect to $u \circ f : S \to \mathbb{A}^1$, so the isomorphism (1.5) implies that we have a natural isomorphism

$$i'^*\mathcal{F}(-\tfrac{1}{2})[-1] \cong i'^!\mathcal{F}(\tfrac{1}{2})[1]$$

where $i'$ is the base change of $i$. Since $i'^*$ has cohomological amplitude $[-1,0]$ while $i'^!$ has cohomological amplitude $[0,1]$ ([BBDG18, Corollaire 4.1.10]) we conclude that $i'^*\mathcal{F}(-\frac{1}{2})[-1] \in {}^p\mathrm{D}^{\leq 0}(Y \times_S S_0)$. Since $g : S' \to S$ factors through $S_0$, we can replace $S$ by $S_0$ (without changing $S'$). Continuing in this way, we reduce to the case where $\dim(S') = \dim(S)$. Now $\mathcal{F}'$ has usual cohomology sheaves only in degrees $\leq -\dim(S)$, so (6.1) holds. □

Finally we present a proof of a result from [BG02].

**Lemma 7** ([BG02, Lemma 7.1.3])**.** *Consider a Cartesian diagram of finite type algebraic stacks*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & S' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & S \end{array}$$

*where $S$ is smooth. Let $j : Y_0 \hookrightarrow Y$ be an open dense substack such that the map $f \circ j : Y_0 \to S$ is smooth. In addition, assume that the complexes $\mathrm{IC}_Y$ and $j_!(\mathrm{IC}_{Y_0})$ are ULA with respect to the map $f$.*

*Denote the closure[2] of $Y_0 \times_S S'$ in $Y'$ by $\overline{Y_0 \times_S S'}$. Then there is a natural isomorphism*

$$\mathrm{IC}_{\overline{Y_0 \times_S S'}} \cong \mathrm{IC}_{S'} \underset{S}{\boxtimes} \mathrm{IC}_Y := f'^*(\mathrm{IC}_{S'}) \otimes g'^*(\mathrm{IC}_Y)[-\dim S],$$

*where the left hand side is implicitly extended by zero to $Y'$.*

*Proof.* Let $Y_0' := Y_0 \times_S S'$ denote the open substack of $Y' = Y \times_S S'$. The uniqueness property of intermediate extensions implies that the lemma amounts to showing:

(i) The $*$-restriction of $\mathrm{IC}_Y \boxtimes_S \mathrm{IC}_{S'}$ to $Y' - Y_0'$ lives in ${}^p\mathrm{D}^{<0}(Y' - Y_0')$.
(ii) The !-restriction of $\mathrm{IC}_Y \boxtimes_S \mathrm{IC}_{S'}$ to $Y' - Y_0'$ lives in ${}^p\mathrm{D}^{>0}(Y' - Y_0')$.
(iii) $\mathrm{IC}_{Y_0} \boxtimes_S \mathrm{IC}_{S'} \cong \mathrm{IC}_{Y_0'}$.

Assertion (iii) is immediate from smoothness of $f \circ j : Y_0 \to S$. The other assertions are local in the smooth topology, so we may assume that all stacks are reduced schemes. Corollary 5 implies that $\mathrm{IC}_Y \boxtimes_S \mathrm{IC}_{S'}$ is Verdier self-dual, so it suffices to check the first assertion. Let $i : Y - Y_0 \hookrightarrow Y$ denote the closed embedding complementary to $j$ and let $i' : Y' - Y_0' \hookrightarrow Y'$ denote its base change. We have a distinguished triangle

$$j_!(\mathrm{IC}_{Y_0}) \to \mathrm{IC}_Y \to i_*i^*(\mathrm{IC}_Y).$$

Since $j_!(\mathrm{IC}_{Y_0})$ and $\mathrm{IC}_Y$ are both assumed to be ULA with respect to $f$, it follows that $i_*i^*(\mathrm{IC}_Y)$ is also ULA with respect to $f$. On the other hand, by proper base change and projection formula, we have a natural isomorphism

$$i_*i^*(\mathrm{IC}_Y) \underset{S}{\boxtimes} \mathrm{IC}_{S'} \cong i'_*i'^*(\mathrm{IC}_Y \underset{S}{\boxtimes} \mathrm{IC}_{S'}),$$

so it suffices to show that the left hand side lives in strictly negative perverse cohomological degrees. By definition of $\mathrm{IC}_Y$, we know that $i_*i^*(\mathrm{IC}_Y) \in {}^p\mathrm{D}^{<0}(Y)$. Therefore, assertion (i) follows from Lemma 6 with $\mathcal{F} = i_*i^*(\mathrm{IC}_Y)(-\frac{1}{2})[-1]$ and $\mathcal{F}' = \mathrm{IC}_{S'}$. □

[2] In general it is possible for $Y'$ to have more irreducible components than $\overline{Y_0 \times_S S'}$.

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