Perverse sheaves and relative Langlands duality

9/17/21

General guiding framework of Ben-Zvi – Sakellaridis – Venkatesh (BZSV)

Number theory $\longleftrightarrow$ **Physics**

Periods and L-functions

S-duality of Boundary Theories in SYM TFT $d=4$ $\mathcal{N}=4$ (the geometric Langlands TFT)

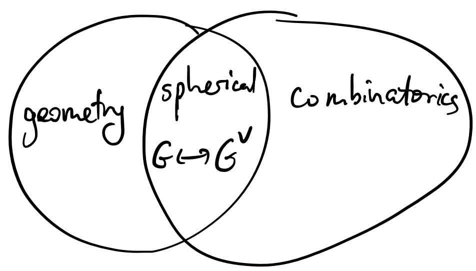
Gaiotto-Witten

Match $G \circlearrowright M \longleftrightarrow M^\vee \circlearrowleft G^\vee$ Hamiltonian manifolds

BZSV: how to go in "one direction" (fix $\mathcal{A}$ and $\mathcal{B}$ twists)

Starting point: $G \circlearrowright X$ smooth affine spherical variety $/\mathbb{C}$, G reductive

spherical $=$ normal variety w/ open B-orbit



Combinatorially, can define

$\check{G}_X$ – spherical dual gp

$\downarrow$ ← finite to 1 map

$\check{G}$

History:
Root system goes back to:
Cartan, Luna-Vust, Brion, Knop
- Gaitsgory-Nadler (Tannakian)
- Sakellaridis-Venkatesh (combinatorial)
Knop-Schalke

$\mathbb{V}_X$: graded super $\check{G}_X$-representation

give highest wts in terms of prime B-divisors of X (combinatorics)

Sakellaridis – Virtual Rep

Sakellaridis-W.

True rep when $\check{G}_X = \check{G}$

$\mathrm{Sym}^\cdot(\mathbb{V}_X)$ formal dg-alg $\leftarrow$ super on $\mathbb{V}_X$ s.t. this is symmetric alg w/o grading

"3d mirror symmetry"

Local Conjecture (BZSV)

(Some technical assumptions on X, e.g. $\check{G}_X \subset \check{G}$)

Hamiltonian spaces $G \circlearrowright T^*X \longleftrightarrow \check{G} \overset{\check{G}_X}{\times} \mathbb{V}_X$

$$\check{G} \overset{\check{G}_X}{\times} \mathbb{V}_X \to \check{\mathfrak{g}}^*$$

There is equiv of categories: $(\check{G} \overset{\check{G}_X}{\times} \mathbb{V}_X)/\check{G}$

$$\shortparallel$$

$$D_c(X(F)/G(O)) \xrightarrow{\sim} D_{perf}(\mathbb{V}_X / \check{G}_X)$$

$$X(F)(\mathbb{C}) = X(F)$$

$F = \mathbb{C}((t))$ $\quad$ $X(F)$ = formal loop space $\leftarrow$ ind-∞-dim'l

$O = \mathbb{C}[[t]]$ $\quad$ $G(O)$ = formal arc space

Want: understand $X \rightsquigarrow \check{G}_X, V_X$ better

How to go $(\check{G}_X, \mathbb{V}_X) \rightsquigarrow X$?

Examples: $X = G \circlearrowleft G \times G$ $\quad$ $\check{G}_X = \check{G} \xrightarrow{\Delta} \check{G} \times \check{G}$ $\quad$ $\mathbb{V}_X = \check{\mathfrak{g}}^*[2]$

$$(\check{G} \times \check{G}) \overset{\check{G}_X}{\times} V_X = T^*(\check{G})$$

Thm (derived Satake, Bezrukavnikov–Finkelberg)

$$D_c(\underbrace{G(O)\backslash G(F)/G(O)}_{\shortparallel \ Gr_G}) = D_{perf}(\check{\mathfrak{g}}^*[2]/\check{G})$$

Extra part of conjecture (boundary of 4d theory)

$$D_c(X(F)/G(\mathcal{O})) \xrightarrow{\sim} D_{perf}\left(\check{G} \overset{\check{G}_X}{\times} \mathbb{V}_X / \check{G}\right)$$

Hecke $\circlearrowleft$ $\qquad$ $\circlearrowleft$ pullback, tensor

$$D_c(G(\mathcal{O})\backslash G(F)/G(\mathcal{O})) \underset{\text{der. Satake}}{\xrightarrow{\sim}} D_{perf}(\check{\mathfrak{g}}^*/\check{G})$$

More examples:

	X	G	$\check{G}_X$	$\mathbb{V}_X$
Whittaker	$(N,\psi)\backslash G$	G	$\check{G}$	$\{0\}$
Mirabolic [BFGT]	GL_n	$GL_n \times GL_{n-1}$	$\check{G}$	$T^*(\mathbb{C}^n \otimes \mathbb{C}^{n-1})_{odd}[1]$
	$GL_n \times \mathbb{C}^n$	$GL_n \times GL_n$	$\check{G}$	$T^*(\mathbb{C}^n \otimes \mathbb{C}^n)_{odd}[1]$

Let's inspect $T^*(GL_n \times \mathbb{C}^n) \longleftrightarrow T^*(\mathbb{C}^n \otimes \mathbb{C}^n) \simeq T^* M_n$

$$\| $$

$$GL_n \times \mathfrak{gl}_n \times \mathbb{C}^n \times \mathbb{C}^{n*}$$

Swap: Thm (Tsao-Hsien Chen – W.)

$$D_c^!(GL_n(\mathcal{O})\backslash M_n(F)/GL_n(\mathcal{O})) \simeq D_{perf}\left(\mathfrak{gl}_n^*[2] \times \mathbb{C}[2] \times \mathbb{C}^* / GL_n\right)$$

What if X singular?

Assume $\check{G} = \check{G}_X$ from now on. $\qquad V_X$ plain $\check{G}$-rep.

Then (Sakellaridis – W.):

can still define $X \rightsquigarrow \mathbb{V}_X = (V_X)_{odd}[1]$

$$D_c(X(F)/G(\mathcal{O})) = \left(\text{probably not so easy to describe}\right)$$

Conjecture Exist equivalence of braided monoidal abelian categories

$$\mathrm{Perv}(X(F)/G(O)) \simeq \Lambda^\cdot(V_X)\text{-mod}^{\check{G}}_{fd}$$

$$s\mathrm{Perv}(X(F)/G(O)) \simeq s\mathrm{Rep}(\underbrace{\check{G} \ltimes (V_X)_{odd}}_{\text{degenerate supergroup}})$$

fusion $*$ $\longleftrightarrow$ $\otimes$

Conj proved in mirabolic case in [BFGT] by Koszul duality (X smooth)

Some evidence for conj:

Fix base point $x_0 \in X$ in open B-orbit

B-action on x_0

$$\begin{tikzcd} & Gr_B \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X(F)/G(O) & & Gr_T \end{tikzcd}$$

Define Jacquet functor $J^! : D(X(F)/G(O)) \longrightarrow D(Gr_T)$

$$J^!(\mathcal{F}) = q_* p^!(\mathcal{F})$$

Factorization

C smooth curve $/\mathbb{C}$

Identify $\mathbb{C}[[t]] = \hat{O}_c$ formal completion at $c \in C$

$(X(F)/G(O))_{Ran}$ "multi-point version"

$$J^! : D((X(F)/G(O))_{Ran}) \longrightarrow D(Gr_{T,Ran})$$

map of factorization categories

Expected: $IC_{X(\mathcal{O})_{Ran}}$ is factorization unit

Then

$$J^{!,enh} : \mathrm{Perv}\left((X(F)/G(\mathcal{O}))_{Ran} \right) \longrightarrow J^{!,enh}(IC_{X(\mathcal{O})_{Ran}})\text{-mod}^{fact}(D(Gr_T)_{Ran})$$

Thm (Sakellaridis – W.)

super Chevalley complex

$$J^{!,enh}(IC_{X(\mathcal{O})_{Ran}}) \cong \mathrm{Fact}\left(C^\cdot(\check{N} \ltimes V^+_{X,odd}, \mathbb{C}) \right)$$

is perverse

graded factorization alg. associated to comm. alg.

(almost — we don't check differential)

There is always a polarization $V_X = T^*(V_X^+)$ as $\check{T}$-reps.

$V_X^+ \in \mathrm{Rep}(\check{B})$

Weights of V_X^+ specified by "X-positive cone"

Guess $J^{!,enh}$ matches

$$\mathrm{Rep}(\check{G} \ltimes V_X) \longrightarrow \mathrm{Rep}(\check{T})$$

$$M \longmapsto C^\cdot(\check{N} \ltimes V_X^+, M) = \underset{\check{N} \ltimes V_X^+}{\mathrm{RHom}}(\mathbb{C}, M)$$

Question Is there q-deformed version of conjecture?

$$\mathrm{Perv}_q(X(F)/G(\mathcal{O})) \simeq \mathrm{Rep}_q(\check{G}_X)$$

for quantum supergroup $U_q(\check{G}_X)$?

True for mirabolic case: $\check{G}_X = GL(n|n)$

q-deformed conjecture proved by [BFT]

For q not root of unity, can hope to use $J^{!,\mathrm{enh}}$ and "quantum doubling" and [BFS]