LOCAL ALGEBRA IN ALGEBRAIC GEOMETRY

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Contents

Introduction

The goal of these notes is to provide an overview of some facts from local algebra, and more importantly, how they relate to algebraic geometry. The content is based on the course Math 233B. Theory of Schemes, taught by Dennis Gaitsgory in Spring 2010 at Harvard[1]. We will try to keep the exposition succinct, citing books for proofs whenever possible. For the local algebra, we will cite mainly [Mat80, Mat89]. We find that [Mat80] is easier to read, but [Mat89] is better typeset and sometimes provides more general results. We will only provide proofs that are not prevalent in the literature, or provide alternative proofs that use more high-tech machinery. In particular, we will assume knowledge of derived categories (cf. [Wei94]).

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[1]See also the lecture-by-lecture notes TeXed by Iurie Boreico.

0.1. **Associated primes.** Let A be a commutative ring and M an A-module.

Definition 0.1.1. A prime ideal $\mathfrak{p}$ of A is an *associated prime ideal* of M if $\mathfrak{p} = \operatorname{ann}(x)$ for some $x \in M$.

The theory of associated primes only really works when A is noetherian.

Proposition 0.1.2. *Let A be noetherian and M a finitely generated*[2] *A-module. Then $\mathfrak{p}$ is an associated prime of M if and only if it is an irreducible component of the support of a submodule of M.*

Proof. See [Mat80, (7.D) Theorem 9, (7.F) Lemma] or [Mat89, Theorem 6.3, Theorem 6.5]. □

Corollary 0.1.3. *If A is noetherian and M finitely generated, then there exists a filtration $0 \subset M_1 \subset \cdots \subset M_n = M$ such that $M_i/M_{i-1} \simeq A/\mathfrak{p}_i$ for primes $\mathfrak{p}_i$, and $\operatorname{Ass} M \subset \{\mathfrak{p}_i\}$.*

Proof. See [Mat80, (7.E) Theorem 10] or [Mat89, Theorem 6.4]. □

Corollary 0.1.4. *If A is noetherian, then a finitely generated A-module has finitely many associated primes.*

Corollary 0.1.5. *If A is noetherian, then the set of zero-divisors of M is the union of all associated primes of M.*

Proof. See [Mat80, (7.B) Corollary 2] or [Mat89, Theorem 6.1(ii)]. □

Corollary 0.1.6. *Let A be noetherian and $\mathfrak{a} \subset A$ an ideal. Then the following are equivalent:*

(1) *$\mathfrak{a}$ is not contained in any associated prime,*
(2) $M^{\mathfrak{a}} := \{x \in M \mid \mathfrak{a}x = 0\} = 0$,
(3) *there exists $f \in \mathfrak{a}$ not a zero-divisor of M.*

Proof. Use prime avoidance and the previous corollary. □

Corollary 0.1.7. *The map $M \to \oplus_{\mathfrak{p}\in \operatorname{Ass}(M)} M_{\mathfrak{p}}$ is injective.*

Proof. If $x \in M$ lies in the kernel, then $\operatorname{ann}(x)$ is not contained in any associated prime. Hence $x = 0$ by Corollary 0.1.6. □

0.2. **Depth.** Let A be a commutative ring, and M an A-module.

Definition 0.2.1. We say $f_1, \dots, f_r \in A$ form an *M-regular sequence* if f_i is not a zero-divisor on $M/(f_1, \dots, f_{i-1})M$ and $M \neq (f_1, \dots, f_r)M$.

0.2.2. *The Koszul complex.* For an element $f \in A$, we define the complex

$$K(f) = (A \xrightarrow{f} A)$$

living in degrees -1 and 0. For $f_1, \dots, f_r \in A$ and an A-module M, define the Koszul complex $K(f_1, \dots, f_r, M)$ to be the tensor product of complexes

$$K(f_1) \underset{A}{\otimes} \cdots \underset{A}{\otimes} K(f_r) \underset{A}{\otimes} M.$$

[2]The proposition is still true if M is not finitely generated, but we replace "irreducible component of" with "minimal element of", but this does not has as nice a geometric interpretation.

Denote the cohomology of this complex by

$$H^i(f_1,\dots,f_r,M) := H^i(K(f_1,\dots,f_r,M)).$$

Then $H^0(f_1,\dots,f_r,M) = M/(f_1,\dots,f_r)M$ and $K(f_1,\dots,f_r,M) \simeq \wedge^\bullet(A^r)\otimes_A M$ can be seen explicitly as

$$0 \to \bigoplus_{i_1<\cdots<i_r} M \to \cdots \to \bigoplus_{i_1<i_2} M \to \bigoplus_{i=1,\dots,r} M \to M.$$

The main point of Koszul complexes is this:

Theorem 0.2.3. *If $f_1,\dots,f_r$ is an M-sequence, then $H^i(f_1,\dots,f_r,M) = 0$ for $i > 0$. Conversely if $(A,\mathfrak{m})$ is a noetherian local ring, $f_1,\dots,f_r \in \mathfrak{m}$, and M is finitely generated, then $H^1(f_1,\dots,f_r,M) = 0$ implies $f_1,\dots,f_r$ is M-regular.*

Proof. See [Mat89, Theorem 16.5]. □

We also mention that besides the Koszul complex, there is another way of thinking about regular sequences when A is noetherian, called quasi-regularity (cf. [Mat80, (15.B)] or [Mat89, Theorem 16.3]).

Definition 0.2.4. Let A be a noetherian ring, M a finitely generated A-module, and $\mathfrak{a} \subset A$ an ideal such that $\mathfrak{a}M \neq M$. Then the $\mathfrak{a}$-depth of M is

$$\operatorname{depth}_{A,\mathfrak{a}}(M) := \inf\{i \mid \operatorname{Ext}^i_A(A/\mathfrak{a}, M) \neq 0\}.$$

(By convention, the depth is ∞ if $\mathfrak{a}M = M$.)

When $(A,\mathfrak{m})$ is a local ring, depth refers to $\operatorname{depth}_{A,\mathfrak{m}}$.

0.2.5. Assume the conditions of Definition 0.2.4 for the rest of this section.

Note that $\operatorname{depth}_{\mathfrak{a}}(M) = 0$ if and only if $M^{\mathfrak{a}} \neq 0$. So Corollary 0.1.6 says that if $\operatorname{depth}_{\mathfrak{a}}(M) > 0$, then there exists $f \in \mathfrak{a}$ that is M-regular.

Lemma 0.2.6. *Suppose $f \in \mathfrak{a}$ is M-regular. Then* $\operatorname{depth}(M) = \operatorname{depth}(M/fM)+1$.

Proposition 0.2.7. *The following are equivalent:*

1. $\operatorname{depth}_{\mathfrak{a}}(M) \geq n$,
2. $\operatorname{Ext}^i_A(N,M) = 0$ *for $i < n$ and any finitely generated A-module N with* $\operatorname{supp}(N) \subset V(\mathfrak{a})$,
3. *there exists a finitely generated A-module N with* $\operatorname{supp}(N) = V(\mathfrak{a})$ *such that* $\operatorname{Ext}^i_A(N,M) = 0$ *for $i < n$,*
4. *for any M-sequence $f_1,\dots,f_i \in \mathfrak{a}$ with $i < n$, there exists $f_{i+1} \in \mathfrak{a}$ extending the regular sequence,*
5. *there exists an M-sequence of length n.*

Proof of lemma and proposition. See [Mat80, (15.C) Theorem 28] or [Mat89, Theorem 16.6]. □

Lemma 0.2.8. *A prime $\mathfrak{p}$ lies in* $\operatorname{Ass}(M)$ *if and only if $M_{\mathfrak{p}}$ has depth 0.*

Proof. Follows from [Mat89, Theorem 6.2]. □

Proposition 0.2.9. *We have* $\operatorname{depth}_{\mathfrak{a}}(M) = \inf\{\operatorname{depth}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in V(\mathfrak{a})\}$.

Proof. See [Mat80, (15.F) Proposition]. □

Lemma 0.2.10. *Let $\varphi : B \to A$ be a homomorphism of noetherian rings and M an A-module that is finitely generated over B. Then for a prime $\mathfrak{q} \in \operatorname{Spec}(B)$,*

$$\operatorname{depth}_{B,\mathfrak{q}}(M) = \inf\{\operatorname{depth}_{A,\mathfrak{p}}(M) \mid \mathfrak{p} \in \operatorname{Spec}(A \underset{B}{\otimes} \kappa(\mathfrak{q}))\}.$$

Proof. Let us first prove the case where A is finitely generated as a B-module. Localizing at $\mathfrak{q}$, we may assume that $(B, \mathfrak{n})$ is a local ring. If $f_1, \ldots, f_r \in \mathfrak{n}$ is M-regular, then $\varphi(f_1), \ldots, \varphi(f_r) \in \mathfrak{m}A_\mathfrak{m}$ is $M_\mathfrak{m}$-regular for any $\mathfrak{m} \in \operatorname{Spec}(A/\mathfrak{n}A)$. This shows the $\leq$ direction.

Conversely, suppose that $\operatorname{depth}_{A_\mathfrak{m}}(M_\mathfrak{m}) \geq r$ for all $\mathfrak{m}$ contracting to $\mathfrak{n}$. Consider the adjunction

$$\operatorname{Ext}^i_B(\kappa(\mathfrak{n}), M) = \operatorname{Hom}_{D(B\text{-mod})}(\kappa(\mathfrak{n}), M[i]) \simeq \operatorname{Hom}_{D(A\text{-mod})}(A \overset{L}{\underset{B}{\otimes}} \kappa(\mathfrak{n}), M[i]).$$

Let $N_j = \operatorname{Tor}^B_j(A, \kappa(\mathfrak{n}))$, which is a finitely generated A-module supported on $\operatorname{Spec}(A/\mathfrak{n}A)$. Then $\operatorname{Ext}^i_A(N_j, M)_\mathfrak{m} \simeq \operatorname{Ext}^i_{A_\mathfrak{m}}((N_j)_\mathfrak{m}, M_\mathfrak{m}) = 0$ for $\mathfrak{m} \in \operatorname{Spec}(A/\mathfrak{n}A)$ and $i < r$ by Proposition 0.2.7; hence $\operatorname{Ext}^i_A(N_j, M) = 0$. Applying $\operatorname{Hom}_{D(A\text{-mod})}$ to the distinguished triangle

$$\tau^{<j}(A \overset{L}{\underset{B}{\otimes}} \kappa(\mathfrak{n})) \to \tau^{\leq j}(A \overset{L}{\underset{B}{\otimes}} \kappa(\mathfrak{n})) \to H^j(A \overset{L}{\underset{B}{\otimes}} \kappa(\mathfrak{n})) = N_j,$$

we deduce that $\operatorname{Hom}_{D(A\text{-mod})}(A \otimes^L_B \kappa(\mathfrak{n}), M[i]) = 0$ since $A \otimes^L_B \kappa(\mathfrak{n}) \in D^-(A\text{-mod})$. Therefore $\operatorname{Ext}^i_B(\kappa(\mathfrak{n}), M) = 0$ for $i < r$, which proves the $\geq$ direction.

Now we drop the assumption that B is finitely generated as an A-module, but we still assume that M is finitely generated as a B-module. By what we have just shown, we can replace $B \to A$ by the inclusion $B/\operatorname{ann}_B(M) \hookrightarrow A/\operatorname{ann}_A(M)$. Then $A \hookrightarrow \operatorname{End}_B(M)$ is an injection, which implies that A is finitely generated as a B-module, bringing us back to the first case. □

Lemma 0.2.11 (Ischebeck)**.** *Let A be a noetherian local ring and M, N non-zero finitely generated A-modules. Then $\operatorname{Ext}^i_A(N, M) = 0$ for $i < \operatorname{depth}(M) - \dim(N)$.*

Proof. See [Mat80, (15.E) Lemma 2] or [Mat89, Theorem 17.1]. □

Theorem 0.2.12. *Let A be a noetherian local ring. Then $\operatorname{depth}(M) \leq \dim(A/\mathfrak{p})$ for any $\mathfrak{p} \in \operatorname{Ass}(M)$. In particular, $\operatorname{depth}(M) \leq \dim(M)$ when $M \neq 0$.*

Proof. See [Mat80, (15.E) Theorem 29] or [Mat89, Theorem 17.2]. □

See §2 for more on depth.

1. Regular local rings and their cohomological properties

1.1. **Notions of dimension.** Let A be a ring (for the moment we allow non-commutativity), and M an A-module (all modules will be left modules).

Definition 1.1.1. The *projective (resp. injective, Tor) dimension* of M is the minimal integer n (or ∞) such that $\operatorname{Ext}^i_A(M, -)$ (resp. $\operatorname{Ext}^i(-, M)$, $\operatorname{Tor}_i(-, M)$) vanishes for all $i > n$.

Lemma 1.1.2. *If $0 \to M' \to P \to M \to 0$ is a short exact sequence of A-modules such that P is projective, then* $\operatorname{proj\,dim}(M') = \operatorname{proj\,dim}(M) - 1$.

Lemma 1.1.3. *The following are equivalent:*

(1) $\operatorname{proj}\dim(M) \leq n$,
(2) *there exists a projective resolution of M of finite length $\leq n$,*
(3) *for an exact sequence $P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ with P_i projective, the* $\ker(P_{n-1} \to P_{n-2})$ *is projective.*

Proof. See [Mat89, Appendix B]. □

The analogous results hold for injective and Tor dimension. Note that projective dimension is at least the Tor dimension.

Definition 1.1.4. The *(left) global or (co)homological dimension* of A is defined by

$$\operatorname{gl}\dim(A) := \sup\{\operatorname{proj}\dim(M) \mid M \text{ is a left } A\text{-module}\}.$$

Proposition 1.1.5. *The following are equivalent:*

(1) $\operatorname{gl}\dim A \leq n$,
(2) $\operatorname{proj}\dim(M) \leq n$ *for every finitely generated A-module M,*
(3) $\operatorname{inj}\dim(N) \leq n$ *for every A-module N.*

If A is (left) noetherian, then the above are also equivalent to:

(3) $\operatorname{Ext}^i_A(M, N) = 0$ $(i > n)$ *for all finitely generated A-modules M, N.*

Proof. See [Mat80, §18] or [Mat89, §19, Lemma 2]. □

Lemma 1.1.6. *Let A be a (left) noetherian ring of finite global dimension. For a finitely generated A-module M, its projective dimension is equal to* $\operatorname{grade}(M)$*, the minimal integer n such that* $\operatorname{Ext}^i_A(M, A) = 0$ *for all $i > n$.*

Proof. Suppose that $\operatorname{grade}(M) = 0$. Let N be an A-module. Since A has finite global dimension, there is a finite projective resolution $P^\bullet \to N$. Since a projective module is a direct summand of a free module and $\operatorname{Ext}^i_A(M, -)$ commutes with direct sums, we deduce that $\operatorname{Ext}^i_A(M, P^n) = 0$ for $i > 0$. Thus $P^\bullet \in K^b(A\text{-mod})$ is an $R\operatorname{Hom}(M, -)$-acyclic resolution of N, so $R\operatorname{Hom}(M, N) \simeq \operatorname{Hom}(M, P^\bullet)$ in $D(\mathbf{Ab})$. Taking the i^{th} cohomology gives $\operatorname{Ext}^i_A(M, N) = 0$ for $i > 0$ since $P^\bullet$ lives in non-positive degrees.

The lemma now easily follows by induction. □

1.2. **Dualizing functor.** Let A be a (left) noetherian ring with $\operatorname{gl}\dim(A) \leq n$. We consider the triangulated subcategory $D^b_{fg}(A\text{-mod}) \subset D(A\text{-mod})$ consisting of complexes with bounded, finitely generated cohomology.

Lemma 1.2.1. *For every $M^\bullet \in D^b_{fg}(A\text{-mod})$, there exists a finite complex $P^\bullet$ consisting of finitely generated projective A-modules and a quasi-isomorphism*

$$P^\bullet \to M^\bullet.$$

Proof. By truncation, we can assume $M^\bullet$ is a finite complex, say living in degrees $[-k, 0]$. Then there exists $P^\bullet \in D^-_{fg}(A\text{-mod})$ with a quasi-isomorphism $P^\bullet \to M^\bullet$. Since $\operatorname{gl}\dim(A) \leq n$, we see that $\ker(P^{-k-n+1} \to P^{-k-n+2})$ will be projective, so we can truncate $P^\bullet$ to prove the claim. □

Corollary 1.2.2. *There is an equivalence of triangulated categories*

$$D^b_{fg}(A\text{-}mod) \leftarrow K^b_{proj,fg}(A\text{-}mod).$$

Proposition 1.2.3. *The derived Hom*

$$\mathbb{D}_{A\to A^{\mathrm{op}}} := R\mathrm{Hom}(-,A) : D^b_{fg}(A\text{-}mod) \to \left(D^b_{fg}(A^{\mathrm{op}}\text{-}mod)\right)^{\mathrm{op}}$$

is a dualizing functor, satisfying $\mathbb{D}_{A^{\mathrm{op}}\to A} \circ \mathbb{D}_{A\to A^{\mathrm{op}}} = \mathrm{id}$.

Proof. By Corollary 1.2.2, we can work with bounded finitely generated projective complexes. For such a complex $P^\bullet = (P^{-k} \to \cdots \to P^0)$, we see that $\mathbb{D}(P^\bullet) = (\check{P}^0 \to \cdots \to \check{P}^{-k})$, where $\check{Q} := \mathrm{Hom}_A(Q,A)$. Everything is now clear since projective modules are direct summands of free modules. □

Remark 1.2.4. Observe from the above proof that if $M^\bullet$ is a complex living cohomologically in degrees $[a,b]$, then $\mathbb{D}(M^\bullet)$ lives cohomologically in degrees $[-b, n-a]$.

Proposition 1.2.5. *There is a natural quasi-isomorphism in the derived category*

$$\mathbb{D}(N) \overset{L}{\underset{A}{\otimes}} M \to R\mathrm{Hom}(N,M)$$

for $N \in D^b_{fg}(A\text{-mod})$ *and* $M \in D(A\text{-mod})$.

Proof. Using Corollary 1.2.2, we can replace N by some $P \in K^b_{proj,fg}(A\text{-mod})$. Now because projective modules are direct summands of free modules, we have a natural isomorphism $\mathrm{Hom}(P,M) \simeq \check{P} \otimes_A M$ (this is even true at the level of bicomplexes). □

1.3. **Local rings.** From now on, all rings will be assumed to be commutative. The cohomological picture simplifies when we consider noetherian local rings. Let $(A, \mathfrak{m}, k)$ be a commutative noetherian local ring.

Proposition 1.3.1. *Let M be a finitely generated A-module. The following are equivalent:*

(1) $\mathrm{Tor}^A_{n+1}(k, M) = 0$,
(2) $\mathrm{Tor}\dim(M) \le n$,
(3) $\mathrm{proj}\dim(M) \le n$.

Proof. See [Mat80, (18.B) Lemma 4] or [Mat89, §19, Lemma 1]. □

Corollary 1.3.2. *We have* $\mathrm{gl}\dim(A) \le n$ *if and only if* $\mathrm{Tor}_{n+1}(k,k) = 0$.

Proof. See [Mat80, (18.B) Theorem 41] or [Mat89, §19, Lemma 1]. □

Theorem 1.3.3 (Auslander and Buchsbaum)**.** *Let A be a noetherian local ring and M a finitely generated A-module with* $\mathrm{proj}\dim(M) < \infty$. *Then*

$$\mathrm{proj}\dim(M) + \mathrm{depth}(M) = \mathrm{depth}(A).$$

Proof. See [Mat89, Theorem 19.1] or [Mat80, §16, Exercise 4]. □

1.4. **Serre's theorem on regular local rings.**

Definition 1.4.1. A local ring $(A, \mathfrak{m}, k)$ is *regular* if it is noetherian and

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim(A).$$

If A is a regular local ring of dimension n, then

(1) there exists an A-regular sequence $f_1, \ldots, f_n \in \mathfrak{m}$,
(2) $\mathrm{depth}(A) = n$,
(3) $\mathrm{Ext}^i(k, A) = 0$ for $i > n$

(cf. [Mat80, (17.F) Theorem 36] or [Mat89, Theorem 14.2]).

Lemma 1.4.2. *A regular local ring is an integral domain.*

Proof. See [Mat89, Theorem 14.3] or [Mat80, (17.F) Theorem 36] (a stronger result is proved here). □

1.4.3. We remark that applying Proposition 1.2.5 to k, M and using the Koszul complex shows that $\mathrm{Tor}_i^A(k,M) \simeq \mathrm{Ext}^{n-i}(k,M)$ (non-canonically) for a regular local ring A of dimension n.

Theorem 1.4.4 (Serre)**.** *A noetherian local ring is regular of dimension n if and only if it has finite global dimension equal to n.*

Proof. The "only if" direction is proved in the standard way using the Koszul complex. The "if" direction can be proved using minimal resolutions (cf. [Mat80, (18.G) Theorem 45] or [Mat89, Theorem 19.2]), but we will provide a more "high-tech" proof due to Gaitsgory.

Let $(A, \mathfrak{m}, k)$ be a noetherian local ring with global dimension n. We wish to show that A is regular. We claim[3] that $\mathrm{depth}(A) \geq n$. Let i be the minimal integer such that $\mathrm{Ext}^i(k,A) \neq 0$. Suppose for the sake of contradiction that $i < n$. Consider $\mathbb{D}(k) \in D(A\text{-mod})$. We have $H^i(\mathbb{D}(k)) = \mathrm{Ext}^i_A(k,A)$, which is a finite dimensional k-vector space. Applying $\mathbb{D}$ again, we have a distinguished triangle

$$\mathbb{D}(\tau^{>i}\mathbb{D}(k)) \to \mathbb{D}(\mathbb{D}(k)) \to \mathbb{D}(H^i(\mathbb{D}(k))[-i])$$

By the assumption on the global dimension, $\tau^{>i}\mathbb{D}(k)$ lives cohomologically in degrees $i+1 \leq n$, so Remark 1.2.4 says $H^j(\mathbb{D}(\tau^{>i}\mathbb{D}(k))) = 0$ for $j \geq n-i$. Since $\mathbb{D}(\mathbb{D}(k)) \simeq k$, the long exact sequence gives

$$H^{n-i}(\mathbb{D}(H^i(\mathbb{D}(k))[-i])) = \mathrm{Ext}^n_A(H^i(\mathbb{D}(k)), A) = 0,$$

which implies $\mathrm{Ext}^n_A(k,A) = 0$. Lemma 1.1.6 implies that $\mathrm{proj}\dim(k) = \mathrm{gl}\dim(A) < n$, a contradiction.

We now prove that A is regular by induction on $\dim(A)$. If $n = 0$, then k is a free A-module, so A must be a field. Now suppose $\mathrm{depth}(A) \geq n > 0$. Then $\mathfrak{m} \notin \mathrm{Ass}(A)$, so by prime avoidance we can choose $f \in \mathfrak{m}$ not contained in $\mathfrak{m}^2$ or any associated prime of A (hence f is A-regular). By the induction hypothesis, it will suffice to show that $A' := A/fA$ also has finite global dimension.

Let N be an A'-module. By (an easy version of) the projection formula, we see that

$$k \overset{L}{\underset{A}{\otimes}} N \simeq (k \overset{L}{\underset{A}{\otimes}} A') \overset{L}{\underset{A'}{\otimes}} N$$

in $D(A'\text{-mod})$. We will show that in this derived category, $k \otimes_A^L A'$ contains k as a direct summand. This would imply by the above formula that $\mathrm{Tor}_i^{A'}(k,N)$ is a direct summand of $\mathrm{Tor}_i^A(k,N)$, proving that k has finite projective dimension as an A'-module.

Since we have a two-step free resolution $A \xrightarrow{f} A$ of A' as an A-module, it follows that $\mathrm{Tor}_i^A(M,A') = 0$ for $i > 1$ and $\mathrm{Tor}_1^A(M,A') = M^f$ for any A-module M. In particular, any A-module without f-torsion is acyclic with respect to $- \otimes_A^L A'$. Hence $\mathfrak{m} \to A$ is an acyclic resolution of k, and we deduce that $k \otimes^L_A A'$ is represented by the complex $(\mathfrak{m} \otimes_A A' \to A')$. If we choose some k-linear map $\mathfrak{m}/\mathfrak{m}^2 \twoheadrightarrow k$ sending

[3] This follows from Theorem 1.3.3, but let us not use it.

the image of f to 1, then the composition $\mathfrak{m} \otimes_A A' \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2 \twoheadrightarrow k$ defines a quasi-isomorphism

$$\left(\mathfrak{m} \underset{A}{\otimes} A' \to A'\right) \to k[1] \oplus k,$$

providing the desired direct sum. □

Corollary 1.4.5. *If A is a regular local ring, then $A_\mathfrak{p}$ is regular for any $\mathfrak{p} \in \operatorname{Spec} A$.*

1.5. **Cohomology of $\mathbb{D}$ when global dimension is finite.**

Definition 1.5.1. A *regular ring* is a noetherian ring such that the localization at every prime (equivalently maximal) ideal is a regular local ring.

Proposition 1.5.2. *Let A be a regular ring of finite dimension and M a finitely generated A-module. Then we have the following:*
1. $\operatorname{codim} \operatorname{supp}(\operatorname{Ext}^i_A(M, A)) \geq i$ *for all i,*
2. $\operatorname{Ext}^i_A(M, A) = 0$ *for $i < \operatorname{codim} \operatorname{supp}(M)$,*
3. *we have equality in (1) when $i = \operatorname{codim} \operatorname{supp}(M)$.*

The codimension of a closed subscheme is the height of the corresponding ideal.

Proof. (1) Take $\mathfrak{p}$ in the support of $\operatorname{Ext}^i_A(M, A)$. Then since M is finitely generated,

$$\operatorname{Ext}^i_A(M, A)_\mathfrak{p} \simeq \operatorname{Ext}^i_{A_\mathfrak{p}}(M_\mathfrak{p}, A_\mathfrak{p}) \neq 0$$

implies that $i \leq \operatorname{gl} \dim(A_\mathfrak{p}) = \dim(A_\mathfrak{p})$.

(2) Since $\operatorname{codim} \operatorname{supp}(M) \leq \operatorname{codim} \operatorname{supp}(M_\mathfrak{p})$, it suffices to prove the claim for a regular local ring $(A, \mathfrak{m}, k)$. We proceed by induction on $n = \dim(A)$. If $n = 0$, there is nothing to prove. Let i be the smallest integer such that $\operatorname{Ext}^i_A(M, A) \neq 0$, and set $N = \operatorname{Ext}^i_A(M, A)$. By the induction hypothesis, $\operatorname{supp}(N) = \operatorname{Ass}(N) = \{\mathfrak{m}\}$, so N is an extension of copies of k. In particular, $\mathbb{D}(k) \simeq k[-n]$ implies that $\mathbb{D}(N)$ lives cohomologically in degree n. We have a non-zero map $N[-i] \to \mathbb{D}(M)$, so applying $\mathbb{D}$ gives a non-zero map

$$M \simeq \mathbb{D}(\mathbb{D}(M)) \to \mathbb{D}(N[-i]).$$

However by what we just remarked, the right hand side lives cohomologically in degree $n - i > 0$, so this map must be zero, a contradiction.

(3) As in (2), we reduce to the case when A is local, and now $\operatorname{supp}(M) = \{\mathfrak{m}\}$. We want to show that $\operatorname{Ext}^n_A(M, A) \neq 0$, where $n = \dim(A)$. If not, then (2) and the definition of global dimension imply that $\mathbb{D}(M)$ is zero in the derived category, a contradiction. □

The proposition shows that $\mathbb{D}(M)$ lives cohomologically between $\operatorname{codim} \operatorname{supp}(M)$ and $\dim(A)$, with the supports of the cohomologies getting smaller as you increase the degree.

2. Depth

2.1. **Property** (S_k).

Let A be a commutative noetherian ring and M a finitely generated A-module. Consider the affine noetherian scheme $X := \operatorname{supp}(M)$. Then we can think of M as an $\mathcal{O}_X$-module, and Lemma 0.2.10 says that this does not affect the depth. (So when working with M we can replace A with $\mathcal{O}_X$.)

Lemma 2.1.1. *Let $Y \subset X$ be a closed subscheme and $U \overset{j}{\hookrightarrow} X$ its complement. Consider the canonical map $M \to Rj_*j^*(M)$ in $D(X) := D(\mathcal{O}_X\text{-mod})$.* [4]

Then the map $M \to j_*j^*(M)$ is injective if and only if $\operatorname{depth}(M_\mathfrak{p}) \geq 1$ *for all $\mathfrak{p} \in Y$. For $k \geq 2$, the map $M \to \tau^{\leq k-2}(Rj_*j^*(M))$ is a quasi-isomorphism if and only if* $\operatorname{depth}(M_\mathfrak{p}) \geq k$ *for all $\mathfrak{p} \in Y$.*

Proof. Let $M' \in D(X)$ denote the cone of $M \to Rj_*j^*(M)$, so we have a distinguished triangle

$$M \to Rj_*j^*(M) \to M'.$$

Since Rj_* is fully faithful, M' has cohomology supported in Y. The long exact sequence tells us that

$$H^{k-2}(M') \simeq \begin{cases} \ker(M \to j_*j^*(M)) & k = 1 \\ \operatorname{coker}(M \to j_*j^*(M)) & k = 2 \\ H^{k-2}(Rj_*j^*(M)) & k \geq 3. \end{cases}$$

We see that the lemma is equivalent to showing, for $k \geq 1$, that M' lives in cohomological degrees $\geq k-1$ if and only if $\operatorname{depth}(M_\mathfrak{p}) \geq k$ for all $\mathfrak{p} \in Y$.

We proceed by induction on $k \geq 1$. Suppose that M' lives in cohomological degrees $\geq k-2$ and $\operatorname{depth}(M_\mathfrak{p}) \geq k-1$ for all $\mathfrak{p} \in Y$. Then $N := H^{k-2}(M')$ is non-zero if and only if $\operatorname{Hom}_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), N_\mathfrak{p}) \neq 0$ for some[5] prime $\mathfrak{p} \in Y$. The base change isomorphism[6] tells us that everything mentioned so far commutes with localization at $\mathfrak{p}$, so we reduce to the case where $(A, \mathfrak{m}, \kappa)$ is a local ring. For any i, adjunction gives $\operatorname{Hom}_{D(X)}(\kappa, Rj_*j^*(M)[i]) = 0$, since $j^*(\kappa) = 0$. Applying $\operatorname{Hom}_{D(X)}(\kappa, -)$ to the distinguished triangle above therefore shows that

$$\operatorname{Hom}_A(\kappa, N) \simeq \operatorname{Hom}_{D(X)}(\kappa, M'[k-2]) \simeq \operatorname{Hom}_{D(X)}(\kappa, M[k-1]) = \operatorname{Ext}^{k-1}_{\mathcal{O}_X}(\kappa, M),$$

which is non-zero if and only if $\operatorname{depth}(M) \leq k-1$. This completes the inductive step, proving the lemma. ∎

In the course of proving the lemma, we also showed the following:

Corollary 2.1.2. *Let $Y = V(\mathfrak{p})$ for a prime $\mathfrak{p}$. Then $M \to \tau^{\leq k-2}(Rj_*j^*(M))$ is a quasi-isomorphism (resp. $M \to j_*j^*(M)$ is injective in the $k = 1$ case)* when localized at $\mathfrak{p}$ *if and only if* $\operatorname{depth}(M_\mathfrak{p}) \geq k$.

Corollary 2.1.3. *Let $k \geq 0$ be an integer. The following are equivalent:*

1. *For any prime $\mathfrak{p} \in X$ such that* $\dim(M_\mathfrak{p}) \geq k$*, the depth of $M_\mathfrak{p}$ is $\geq k$.*
2. *For any open $U \overset{j}{\hookrightarrow} X$ whose complement has codimension $\geq k$, the map $M \to \tau^{\leq k-2}(Rj_*j^*(M))$ is a quasi-isomorphism (resp. $M \to j_*j^*(M)$ is injective for $k = 1$).*
3. *For any countably generated $\mathcal{O}_X$-module N with* $\operatorname{codim}(\operatorname{supp}(N), X) \geq k$*,* [7] *we have* $\operatorname{Ext}^i_{\mathcal{O}_X}(N, M) = 0$ *for $i < k$.*

---

[4] Since X is an affine noetherian scheme, we do not need to worry about distinguishing between the derived categories of $\mathcal{O}_X$-modules and quasi-coherent sheaves. Here $Rj_* : D(U) \to D(X)$ is the right adjoint of j^*, and it is fully faithful since U is a subspace of X.

[5] Any prime $\mathfrak{p}$ minimal in $\operatorname{supp}(N)$ will do.

[6] See [FGI+05, Theorem 8.3.2]; the key is that U is quasi-compact.

[7] We say that a subset Y of X is of codimension $\geq k$ if for every prime $\mathfrak{p} \in Y$, the height of $\mathfrak{p}$ is $\geq k$. Here the condition "countably generated" can be removed if we require $\operatorname{ann}(N)$ to have height $\geq k$. We do not know if this last requirement is necessary.

Proof. The implication (3) $\Rightarrow$ (1) is evident: for a prime $\mathfrak{p}$, take N to be $A/\mathfrak{p}$. The equivalence of (1) $\Leftrightarrow$ (2) follows immediately from Lemma 2.1.1. Let's show the implication (2) $\Rightarrow$ (3). First, assume that N is finitely generated. Let $U \hookrightarrow X$ be the complement of $\operatorname{supp}(N)$. Then

$$\begin{aligned}\operatorname{Ext}^i_{\mathcal{O}_X}(N, M) &\simeq \operatorname{Hom}_{D(X)}(N, \tau^{\leq k-2}(Rj_*j^*(M))[i]) \\ &\simeq \operatorname{Hom}_{D(X)}(N, Rj_*j^*(M)[i]) \simeq \operatorname{Hom}_{D(U)}(j^*(N), j^*(M)[i]) = 0,\end{aligned}$$

since $i < k-1$ and $j^*(N) = 0$ (the first isomorphism is an injection when $k = 1$). Now suppose N is countably generated, i.e., $N = \varinjlim N_j$ for an exhaustive chain $N_0 \subset N_1 \subset \cdots$ of finitely generated submodules. Let $M \to I^\bullet$ be an injective resolution in $D(X)$. Then $\operatorname{Hom}_{\mathcal{O}_X}(N, I^n) \simeq \varprojlim \operatorname{Hom}_{\mathcal{O}_X}(N_j, I^n)$, where the transition maps are surjective since I^n is injective. In particular, the inverse system of complexes

$$\{\operatorname{Hom}_{\mathcal{O}_X}(N_j, I^\bullet)\}_{j\in\mathbf{N}}$$

satisfies the Mittag-Leffler condition. Upon taking cohomologies, the claim in the finitely generated case together with [Wei94, Theorem 3.5.8] shows that we have $\operatorname{Ext}^i_{\mathcal{O}_X}(N, M) = 0$ for $i < k$. □

Definition 2.1.4. Let $k \geq 0$ be an integer. We say that M has property (S_k) if for every prime $\mathfrak{p}$ of A the module $M_\mathfrak{p}$ has depth at least $\min\{k, \dim(M_\mathfrak{p})\}$.

2.1.5. *Examples.* The condition (S_0) is trivial. The condition (S_1) holds if and only if M has no embedded primes (i.e., the associated primes are the irreducible components of the support of M). A ring A satisfies (R_0) and (S_1) if and only if A is reduced. A regular ring satisfies (R_k) and (S_k) for all $k \geq 0$. Observe that if M satisfies (S_k), then it also satisfies the equivalent conditions of Corollary 2.1.3.

Proposition 2.1.6. *Assume that M satisfies (S_1). Then the image of*

$$M \hookrightarrow \bigoplus_{\mathfrak{p}\in\operatorname{Ass}(M)} M_\mathfrak{p}$$

consists of the elements $(m_\mathfrak{p} \in M_\mathfrak{p})_{\mathfrak{p}\in\operatorname{Ass}(M)}$ that satisfy the following property: for every prime $\mathfrak{q}$ such that $\operatorname{depth}(M_\mathfrak{q}) = 1$, we have that $(m_\mathfrak{p})_{\mathfrak{q}\in V(\mathfrak{p})}$ lies in the image of the natural map $M_\mathfrak{q} \to \oplus_{\mathfrak{q}\in V(\mathfrak{p})} M_\mathfrak{p}$.

Proof. One direction is clear; we prove the other. Take an element $(m_\mathfrak{p})$ satisfying the declared property. Let $i_\mathfrak{p} : \operatorname{Spec}(\mathcal{O}_{X,\mathfrak{p}}) \hookrightarrow X$. Consider the collection $\mathcal{U}$ of open subsets $U \subset X$ such that $(\mathfrak{m}_\mathfrak{p})$ lies in the image of $\Gamma(U, M) \to \oplus\Gamma(U, i_{\mathfrak{p}*}i^*_\mathfrak{p}(M))$. Then Corollary 0.1.7 implies that there is a unique maximal open (the union) $U \in \mathcal{U}$. The condition on $(m_\mathfrak{p})$ implies that U contains all primes $\mathfrak{q}$ with $\operatorname{depth}(M_\mathfrak{q}) \leq 1$, so Lemma 2.1.1 says that $M \simeq j_*j^*(M)$, and we are done. □

2.2. **Serre's criterion of normality.** First, we define normality:

Definition 2.2.1. A ring A is *normal* if for every prime $\mathfrak{p}$ of A the localization $A_\mathfrak{p}$ is an integrally closed domain.

A normal ring is a direct sum of integrally closed domains (cf. [Mat89, pg. 64, Remark]). In particular, an integral domain is normal if and only if it is integrally closed.

2.2.2. For the rest of this section, A will be a commutative noetherian ring.

Theorem 2.2.3. *Let A be a normal noetherian domain. Then $A = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$.*

Proof. See [Mat80, (17.H) Theorem 38] or [Mat89, Theorem 11.5]. □

Definition 2.2.4. We say that A has property (R_k) if for every prime $\mathfrak{p}$ of height $\leq k$ the local ring $A_{\mathfrak{p}}$ is regular.

Theorem 2.2.5 (Serre)**.** *A noetherian ring A is normal if and only if it has property (R_1) and (S_2).*

Proof. See [Mat80, (17.I) Theorem 39] or [Mat89, Theorem 23.8]. We will give a different proof of the implication $(R_1) + (S_2) \Rightarrow$ normal.

Assume that A satisfies (R_1) and (S_2). Then it is reduced, and for every prime $\mathfrak{p}$ of height ≤ 1, the localization $A_{\mathfrak{p}}$ is a domain by Lemma 1.4.2. Thus the union Y of the pairwise intersections of distinct irreducible components of $X := \operatorname{Spec} A$ has codimension ≥ 2; Corollary 2.1.3 implies that A is a direct sum of integral domains. Hence we reduce to the case where A is a domain.

Let K be the field of fractions of A. Let $x \in K$ be integral over A and consider the extension ring $B := A[x] \subset K$, which is finite as an A-module. Let $N := B/A$ so we have an extension $0 \to A \to B \to N \to 0$ of A-modules. Since a DVR is integrally closed, we have that $A_{\mathfrak{p}} = B_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of height ≤ 1. Thus $\operatorname{codim} \operatorname{supp}(N) \geq 2$. Now Corollary 2.1.3 implies that $\operatorname{Ext}^1_A(N, A) = 0$. Therefore $B \simeq A \oplus N$, but B is contained in the fraction field of A, so N must be zero. We conclude that A is integrally closed. □

2.3. **Cohen-Macaulay modules.**

Definition 2.3.1. Let A be a noetherian local ring and M a finitely generated A-module. We say that M is a *Cohen-Macaulay* (abbreviated CM) *module* if $M = 0$ or $\operatorname{depth}(M) = \dim(M)$. If A itself is CM as an A-module then we call A a *Cohen-Macaulay ring.*

By Theorem 0.2.12, we know that $\operatorname{depth}(M) \leq \dim(M)$ in general. So as an example, if A is artinian, then any finitely generated A-module is CM. Also note that a regular ring is CM.

Theorem 2.3.2. *Let $(A, \mathfrak{m})$ be a noetherian local ring and M a finitely generated A-module.*

1. *If M is CM and $\mathfrak{p} \in \operatorname{Ass}(M)$, then $\operatorname{depth}(M) = \dim(A/\mathfrak{p})$.*
2. *If $f_1, \dots, f_r$ is an M-regular sequence in $\mathfrak{m}$, then M is CM if and only if $M/(f_1, \dots, f_r)M$ is CM.*
3. *If M is CM then $M_{\mathfrak{p}}$ is a CM module over $A_{\mathfrak{p}}$ for every prime $\mathfrak{p}$, and if $M_{\mathfrak{p}} \neq 0$ then $\operatorname{depth}_{A,\mathfrak{p}}(M) = \operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$.*

Proof. See [Mat80, (16.A) Theorem 30] or [Mat89, Theorem 17.3]. □

Theorem 2.3.3. *Let $(A, \mathfrak{m})$ be a CM local ring.*

1. *For a proper ideal $\mathfrak{a}$ of A we have*
$$\mathrm{ht}(\mathfrak{a}) = \operatorname{depth}_{\mathfrak{a}}(A) \quad \textit{and} \quad \mathrm{ht}(\mathfrak{a}) + \dim(A/\mathfrak{a}) = \dim(A).$$
2. *A is catenary.*
3. *For any sequence $f_1, \dots, f_r \in \mathfrak{m}$, the following are equivalent:*

(a) $f_1, \ldots, f_r$ *is* A*-regular,*
(b) $\mathrm{ht}(f_1, \ldots, f_i) = i$ *for* $1 \le i \le r$*,*
(c) $\mathrm{ht}(f_1, \ldots, f_r) = r$*,*
(d) $f_1, \ldots, f_r$ *is part of a system of parameters of* A*.*

Proof. See [Mat80, (16.B) Theorem 31] or [Mat89, Theorem 17.4]. □

Proposition 2.3.4. *Let* A *be a regular local ring of dimension* n *and* M *a finitely generated* A*-module. Then* M *is CM of dimension* n *if and only if it is free. More generally,* M *is CM of dimension* k *if and only if* $\mathbb{D}(M)$ *has nonzero cohomology only in degree* $n-k$*.*

Proof. Since A is regular and hence CM, Theorem 2.3.3(1) implies that

$$\operatorname{codim} \operatorname{supp}(M) = \dim(A) - \dim(M).$$

The claim now follows from Lemma 1.1.6, Theorem 1.3.3, and Proposition 1.5.2. □

Definition 2.3.5. Let A be a noetherian ring and M a finitely generated A-module. We say that M is *Cohen-Macaulay* if the localization $M_{\mathfrak{p}}$ is CM for every prime ideal $\mathfrak{p}$ of A. By Theorem 2.3.2 this is equivalent to saying that $M_{\mathfrak{m}}$ is CM for every maximal ideal $\mathfrak{m}$ of A.

Proposition 2.3.6. *Let* $B \to A$ *be a homomorphism of noetherian rings and* M *an* A*-module that is finitely generated over* B*. Then* M *is CM over* A *if and only if it is CM over* B*.*

Proof. As in the proof of Lemma 0.2.10, we may assume that $B \hookrightarrow A$ is an inclusion and A is integral over B. The claim follows from Lemma 0.2.10 and the going-up theorem (cf. [Mat80, (5.E) Theorem 5] or [Mat89, Theorems 9.3-4]). □

Corollary 2.3.7. *Let* k *be a field,* A *a finitely generated* k*-algebra, and* M *a finitely generated* A*-module. Then* M *is CM with* $\operatorname{supp}(M)$ *equidimensional* [8] *of dimension* n *if and only if for any finitely generated* k*-algebra* B *which is regular and equidimensional of dimension* n *with a map* $B \to A$ *such that* M *is finitely generated as a* B*-module,* M *is locally free over it. Moreover, such a* B *always exists.*

Proof. The equivalence of the two statements follows from Propositions 2.3.4 and 2.3.6. By Noether normalization, there exists an inclusion $B \hookrightarrow A/\operatorname{ann}(M)$ where $B := k[x_1, \ldots, x_n]$ satisfies the necessary conditions. □

3. Divisors and line bundles

3.1. **Weil divisors.** Let X be a noetherian integral scheme which is regular in codimension one (i.e., every local ring $\mathcal{O}_{X,x}$ of dimension one is regular). Let K denote the field of fractions of X.

Definition 3.1.1. A *prime divisor* on X is a closed integral subscheme of codimension one. A *Weil divisor* is an element of the free abelian group $\mathrm{Div}^W(X)$ generated by the prime divisors. A Weil divisor $D = \Sigma\, n_Y Y$ is *effective* if $n_Y \ge 0$ for all prime divisors Y.

[8] Recall that since k is universally catenary, a scheme X locally of finite type over k is equidimensional of dimension n if and only if $\dim(\mathcal{O}_{X,x}) = n$ for all closed points $x \in X$ (cf. [Mat89, Theorem 15.6]).

There is a well-defined map of abelian groups $K^* \to \mathrm{Div}^W(X)$ sending f to the *divisor of* f, defined by

$$(f) = \Sigma\, v_Y(f) \cdot Y$$

(cf. [Har77, II, Lemma 6.1]). Such a divisor is said to be *principal*.

Lemma 3.1.2. *If X is normal, we have an exact sequence of abelian groups*

$$0 \to \Gamma(X, \mathcal{O}_X)^* \to K^* \to \mathrm{Div}^W(X).$$

Proof. Note that $\Gamma(U, \mathcal{O}_X^*) := \Gamma(U, \mathcal{O}_X)^*$ defines a *sheaf* of groups $\mathcal{O}_X^*$ on X. This allows us to reduce to the case where $X = \mathrm{Spec}(A)$ is affine. Then $(f) = 0$ implies that $f, f^{-1} \in \bigcap_{\mathrm{ht}(\mathfrak{p})=1} A_\mathfrak{p} = A$ (using Theorem 2.2.3). □

Definition 3.1.3. The *(Weil) divisor class group of* X, denoted $\mathrm{Cl}^W(X)$, is the quotient of $\mathrm{Div}^W(X)$ by the subgroup of principal divisors.

3.2. **Cartier divisors.** Let X be an arbitrary scheme.

Definition 3.2.1. An *effective Cartier divisor* on X is an equivalence class in the set $\mathrm{Div}^{C,+}(X)$ of pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module (i.e., line bundle on X) and $s : \mathcal{O}_X \hookrightarrow \mathcal{L}$ is an injective section, where $(\mathcal{L}, s) \sim (\mathcal{L}', s')$ if there is an isomorphism $\varphi : \mathcal{L} \to \mathcal{L}'$ such that $s' = \varphi \circ s$.

Since an automorphism of $\mathcal{L}$ corresponds to an element of $\Gamma(X, \mathcal{O}_X)^*$, the isomorphism φ above is necessarily unique. We give $\mathrm{Div}^{C,+}(X)$ the structure of an abelian monoid as follows: for two effective divisors $(\mathcal{L}, s), (\mathcal{L}', s')$, the tensor product

$$\left(\mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}',\, \mathcal{O}_X \overset{s \otimes s'}{\hookrightarrow} \mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}'\right) \in \mathrm{Div}^{C,+}(X)$$

has an injective section by flatness of line bundles.

Here is an equivalent definition of effective Cartier divisors: note that an injection $\mathcal{O}_X \hookrightarrow \mathcal{L}$ is the same as an injection $\mathcal{L}^{-1} \hookrightarrow \mathcal{O}_X$ (by applying $- \otimes \mathcal{L}^{-1}$). Now each such equivalence class is represented by an ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_X$, where D is the corresponding closed subscheme of X. Thus effective Cartier divisors are the closed subschemes of X with invertible ideal sheaves.

Define the sheaf[9] of abelian monoids $\mathcal{D}iv_X^{C,+}$ on X by $U \mapsto \mathrm{Div}^{C,+}(U)$ for an open subset $U \subset X$. Let $\mathcal{D}iv_X^C$ be the group completion of $\mathcal{D}iv_X^{C,+}$ in the category of sheaves: this is the sheaf associated to the presheaf sending U to the group completion of $\mathrm{Div}^{C,+}(U)$ (☠ this presheaf might not be a sheaf!).

Definition 3.2.2. The *Cartier divisors* $\mathrm{Div}^C(X)$ on X is the group $\Gamma(X, \mathcal{D}iv_X^C)$.

This definition is a little hard to work with, so we provide an alternative. Consider the sub-sheaf of monoids $\mathcal{S}(\mathcal{O}_X) \subseteq \mathcal{O}_X$ whose sections on U consist of the *regular* elements in $\Gamma(U, \mathcal{O}_X)$ (i.e., the sections such that the induced maps $\mathcal{O}_U \to \mathcal{O}_U$ are injective). This is a sheaf and not simply a presheaf because injectivity is local. Let $\mathcal{M}_X$ denote the sheaf associated to the presheaf $U \mapsto \Gamma(U, \mathcal{O}_X)[\Gamma(U, \mathcal{S}(\mathcal{O}_X))^{-1}]$. For any affine subscheme $U \subset X$, the sections $\Gamma(U, \mathcal{M}_X)$ form the total quotient ring of $\Gamma(U, \mathcal{O}_U)$, but simply taking total quotient rings for arbitrary open subsets U does not even form a presheaf. Observe that $\mathcal{M}_X^*$ is the sheaf group completion of $\mathcal{S}(\mathcal{O}_X)$.

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[9]The gluing axiom is satisfied because of the uniqueness of the isomorphisms in the equivalence relations noted earlier.

Proposition 3.2.3. *There are natural isomorphisms*

1. $\mathcal{S}(\mathcal{O}_X)/\mathcal{O}_X^* \to \mathcal{D}iv_X^{C,+}$ *of sheaves of abelian monoids, and*
2. $\mathcal{M}_X^*/\mathcal{O}_X^* \to \mathcal{D}iv_X^C$ *of sheaves of abelian groups.*

Proof. (1) We define a map $\mathcal{S}(\mathcal{O}_X) \to \mathcal{D}iv_X^{C,+}$ by sending a regular section $s \in \Gamma(U, \mathcal{O}_U)$ to $(\mathcal{O}_U, s) \in \mathrm{Div}^{C,+}(U)$. By definition of the equivalence relation on effective Cartier divisors, we have a kernel sequence

$$0 \to \mathcal{O}_X^* \to \mathcal{S}(\mathcal{O}_X) \to \mathcal{D}iv_X^{C,+},$$

which induces an injective map $\mathcal{S}(\mathcal{O}_X)/\mathcal{O}_X^* \hookrightarrow \mathcal{D}iv_X^{C,+}$. For any $(\mathcal{L}, s) \in \mathrm{Div}^{C,+}(U)$, we can refine U to assume $\mathcal{L} \simeq \mathcal{O}_U$ since $\mathcal{L}$ is a line bundle. This shows surjectivity as sheaves. (2) follows from (1) by taking group completions. □

Finally, we give what is probably the easiest way to think about Cartier divisors. Define a *fractional ideal* on X to be an $\mathcal{O}_X$-submodule of $\mathcal{M}_X$. The functor sending an open subset $U \subset X$ to the abelian group of invertible fractional ideals on U defines a sheaf $\mathcal{I}d.inv_X$. We can define a map of sheaves $\mathcal{D}iv_X^C \to \mathcal{I}d.inv_X$ by locally sending $f \in \Gamma(U, \mathcal{M}_X^*)$ to $\mathcal{O}_U f$.

Proposition 3.2.4. *The map* $\mathcal{D}iv_X^C \to \mathcal{I}d.inv_X$ *is an isomorphism.*

Proof. See [GD67, Proposition 21.2.6]. □

In particular, $\mathrm{Div}^C(X)$ is isomorphic to the group of invertible fractional ideals on X. For a Cartier divisor D on X, we define the invertible fractional ideal $\mathcal{O}_X(D)$ such that $\mathcal{O}_X(D)^{-1}$ is the image of D under the isomorphism of Proposition 3.2.4. The inclusion $\mathcal{O}_X(D) \subset \mathcal{M}_X$ induces an isomorphism

$$s_D^{-1} : \mathcal{O}_X(D) \underset{\mathcal{O}_X}{\otimes} \mathcal{M}_X \to \mathcal{M}_X.$$

Conversely, suppose we are given a line bundle $\mathcal{L}$ on X and a meromorphic trivialization $s^{-1} : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}_X \simeq \mathcal{M}_X$. Then on any open subset $U \subset X$ where $\mathcal{L} \simeq \mathcal{O}_U$ we have that $s|_U$ corresponds to an element of $\Gamma(U, \mathcal{M}_X^*/\mathcal{O}_X^*)$. These glue to give a divisor $\mathrm{div}(s) \in \mathrm{Div}^C(X)$.

Proposition 3.2.5. *The maps*

$$D \mapsto (\mathcal{O}_X(D), s_D), \qquad (\mathcal{L}, s) \mapsto \mathrm{div}(s)$$

give mutually inverse isomorphisms between $\mathrm{Div}^C(X)$ *and the group of equivalence classes of pairs* $(\mathcal{L}, s)$.

The proposition essentially brings us back to the very first definition we gave of effective Cartier divisors.

3.2.6. By Proposition 3.2.3, we have a short exact sequence of sheaves of abelian groups $0 \to \mathcal{O}_X^* \to \mathcal{M}_X^* \to \mathcal{D}iv_X^C \to 0$. By applying sheaf cohomology, this gives a long exact sequence of abelian groups

$$0 \to \Gamma(X, \mathcal{O}_X^*) \to \Gamma(X, \mathcal{M}_X^*) \to \mathrm{Div}^C(X) \to H^1(X, \mathcal{O}_X^*) \to H^1(X, \mathcal{M}_X^*).$$

Definition 3.2.7. The *(Cartier) divisor class group of* X, denoted $\mathrm{Cl}^C(X)$, is the quotient of $\mathrm{Div}^C(X)$ by the image of $\Gamma(X, \mathcal{M}_X^*)$. The *Picard group* $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles on X.

By [Har77, III, Ex. 4.5], we have an isomorphism $H^1(X, \mathcal{O}_X^*) \simeq \mathrm{Pic}(X)$. If X is an integral scheme, then $\mathcal{M}_X^*$ is the constant sheaf with value K^*. A constant sheaf is flasque, so $H^1(X, \mathcal{M}_X^*) = 0$.

Corollary 3.2.8. *If X is integral, we have an isomorphism $\mathrm{Cl}^C(X) \simeq \mathrm{Pic}(X)$.*

3.3. **Relating Weil and Cartier divisors.** As in §3.1, let X be a noetherian integral scheme which is regular in codimension one, so that we can talk about both Weil and Cartier divisors on X. Then $\mathcal{M}_X$ is the constant sheaf K. Define the homomorphism

$$\mathrm{Div}^C(X) \to \mathrm{Div}^W(X)$$

by sending an invertible fractional ideal $\mathcal{I} \subset K$ on X to $\Sigma_Y v_Y(\mathcal{I})$. For an open covering $\{U_i\}$ of X where $\mathcal{I}|_{U_i} = \mathcal{O}_{U_i} f_i$, we have $v_Y(\mathcal{I}) = v_Y(f_i)$.

Proposition 3.3.1. *Let X be as in §3.1. Then the following are equivalent:*

1. *X is normal,*
2. *for any open $X^\circ \subset X$, the map $\mathrm{Div}^C(X^\circ) \to \mathrm{Div}^W(X^\circ)$ is injective,*
3. *for any open $X^\circ \subset X$ and $Y \subset X^\circ$ a closed subscheme of codimension ≥ 2 in X°, a line bundle on the complement $U := X^\circ - Y$ admits at most one extension to X° (i.e., $\mathrm{Pic}(X^\circ) \to \mathrm{Pic}(U)$ is injective).*

Proof. The implication (1) $\Rightarrow$ (2) follows from Theorem 2.2.3. Let us show the implication (1) $\Rightarrow$ (3). Suppose X is normal and let $\mathcal{L}$ be a line bundle on X°. Then $\mathcal{L}$ satisfies (S_2), and Corollary 2.1.3 implies that $\mathcal{L} \simeq j_* j^*(\mathcal{L})$ where j is the inclusion $U \hookrightarrow X^\circ$.

Next we show that (3) $\Rightarrow$ (2). Suppose two invertible fractional ideals $\mathcal{I}, \mathcal{I}'$ on X° have the same Weil divisor. Then there exists $U \subset X^\circ$ containing all points of codimension 1 such that $\mathcal{I}|_U = \mathcal{I}'|_U$ inside $\mathcal{M}_U$. Assumption (3) implies that there is an isomorphism $\mathcal{I} \simeq \mathcal{I}'$, and this must be an equality by uniqueness.

We now prove (2) $\Rightarrow$ (1). We may assume that $X = \mathrm{Spec}(A)$ is affine, where A is an integral domain satisfying (R_1). Suppose there exists $x \in K$ that is integral over A such that $A \subsetneq B := A[x]$. Let $U \subset X$ be the open subset of primes $\mathfrak{p}$ of A such that $A_\mathfrak{p} = B_\mathfrak{p}$. Since DVRs are integrally closed, the complement Y has codimension ≥ 2. Let $\mathfrak{p}$ be an irreducible component of Y. By Nakayama's lemma, the embedding $\kappa(\mathfrak{p}) \hookrightarrow B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is proper. Therefore there exists a meromorphic function $f \in (B_\mathfrak{p})^* \subset K$ that is not in $(A_\mathfrak{p})^*$. Let $X^\circ \subset X$ be an affine open neighborhood of $\mathfrak{p}$ that does not intersect $V(\mathfrak{q})$ for any height 1 prime $\mathfrak{q}$ with nonzero $v_\mathfrak{q}(f)$. Replacing X by X°, we have a meromorphic function $f \in K$ such that $(f) = 0$ in $\mathrm{Div}^W(X)$ and $f \notin A^*$. Therefore Af defines an invertible fractional ideal on X that is trivial as a Weil divisor but non-trivial as a Cartier divisor, contradicting the hypothesis of (2). We conclude that $A = B$, i.e., A is integrally closed. □

3.3.2. *Unique factorization domains.* As we will see, the comparison of Weil and Cartier divisors naturally leads to considering UFDs and locally factorial schemes.

Proposition 3.3.3. *A noetherian domain is a UFD if and only if every prime ideal of height 1 is principal.*

Proof. See [Mat80, (19.A) Theorem 47] or [Mat89, Theorem 20.1]. □

Proposition 3.3.4. *Let A be a noetherian domain. Then A is a UFD if and only if A is normal and $\mathrm{Cl}^W(A) = 0$.*

Proof. See [Mat89, pg. 165]. □

Theorem 3.3.5. *Let X be a normal noetherian scheme. Then the following are equivalent:*

1. *X is locally factorial (i.e., $\mathcal{O}_{X,x}$ is a UFD for every $x \in X$),*
2. *$\mathrm{Div}^C(X) \to \mathrm{Div}^W(X)$ is an isomorphism,*
3. *for every prime divisor Y on X, the corresponding ideal sheaf $\mathcal{I}_Y$ is invertible (i.e., every prime divisor is an effective Cartier divisor).*

Proof. See [GD67, Théorème 21.6.9]. Proposition 3.3.3 shows that (1) ⇔ (3). The implication (3) ⇒ (2) is clear given Proposition 3.3.1. Let us show (2) ⇒ (3). Let Y be a prime divisor on X. By condition (2) and Theorem 2.2.3, there exists an invertible ideal $\mathcal{L} \subset \mathcal{O}_X$ mapping to Y in $\mathrm{Div}^W(X)$. By considering valuations, one sees that $\mathcal{L} \subset \mathcal{I}_Y$. To show that $\mathcal{L} = \mathcal{I}_Y$, we may assume $X = \mathrm{Spec}(A)$ is affine. Then $\mathcal{T} := \mathcal{I}_Y/\mathcal{L}$ is supported in codimension ≥ 2, so Corollary 2.1.3 implies that $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}, \mathcal{L}) = 0$. Since $\mathcal{O}_X$ is torsion-free, we have $\mathcal{T} = 0$. □

Theorem 3.3.6. *Let X be as in §3.1. Then the following are equivalent:*

1. *X is locally factorial.*
2. *for any open $X^\circ \subset X$, the map $\mathrm{Div}^C(X^\circ) \to \mathrm{Div}^W(X^\circ)$ is an isomorphism,*
3. *for any open $X^\circ \subset X$ and $Y \subset X^\circ$ a closed subscheme of codimension ≥ 2 in X°, a line bundle on the complement $U := X^\circ - Y$ admits a unique extension to X° (i.e., $\mathrm{Pic}(X^\circ) \to \mathrm{Pic}(U)$ is an isomorphism).*

Proof. Proposition 3.3.1 and Theorem 3.3.5 show the equivalence (1) ⇔ (2). If X is locally factorial, then we have the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Div}^W(X^\circ) & \twoheadrightarrow & \mathrm{Cl}^W(X^\circ) & \xleftarrow{\sim} & \mathrm{Cl}^C(X^\circ) & \xrightarrow{\sim} & \mathrm{Pic}(X^\circ) \\ \downarrow{\sim} & & & & & & \downarrow \\ \mathrm{Div}^W(U) & \twoheadrightarrow & \mathrm{Cl}^W(U) & \xleftarrow{\sim} & \mathrm{Cl}^C(U) & \xrightarrow{\sim} & \mathrm{Pic}(U) \end{array}$$

where $\mathrm{Div}^W(X^\circ) \simeq \mathrm{Div}^W(U)$ since Y has codimension ≥ 2. This shows an extension always exists, and uniqueness follows from Proposition 3.3.1, so we have proved that (1) ⇒ (3).

We now show the implication (3) ⇒ (1). Proposition 3.3.1 says that X is normal. Let D be a prime divisor on X and let $U \overset{j}{\hookrightarrow} X$ be the open subset of points $x \in X$ where the localization $\mathcal{I}_{D,x}$ is principal. Then the complement of U is contained in D and has codimension ≥ 2. By condition (3), there exists a line bundle $\mathcal{L}$ on X such that $j^*(\mathcal{L}) \simeq j^*(\mathcal{I}_D)$. Replacing $\mathcal{L}$ by its image under $\mathcal{L} \simeq j_*j^*(\mathcal{L}) \simeq j_*j^*(\mathcal{I}_D) \subset K$, we can assume that $\mathcal{L}$ is an invertible fractional ideal. We deduce that $\mathcal{L} = \mathcal{I}_D$ as in the proof of Theorem 3.3.5. □

3.3.7. A standard application of Weil/Cartier divisors is the fact that for a field k, we have $\mathbf{Z} \simeq \mathrm{Pic}(\mathbf{P}^n_k)$ with $1 \mapsto \mathcal{O}(1)$.

Proposition 3.3.8. *Let S be a normal noetherian scheme. Then $S \times \mathbf{A}^1$ is also normal, and the natural map $\mathrm{Pic}(S) \to \mathrm{Pic}(S \times \mathbf{A}^1)$ induced by projection is an isomorphism.*

Proof. See [Bou89, VII, §1.10, Proposition 18]. For completeness, we give a proof of the isomorphism $\mathrm{Pic}(S) \simeq \mathrm{Pic}(S \times \mathbf{A}^1)$.

Let $S \times \mathbf{A}^1 \stackrel{\mathrm{pr}_1}{\to} S$ be the projection onto the first coordinate. Since pr_1 admits a section, pr_1^* is injective. We will show surjectivity when S is normal.

By considering connected components, we may assume that S is integral. Let K denote the field of fractions of S. Then $S \times \mathbf{A}^1$ is integral with field of fractions $K(X)$ for an indeterminate X. Therefore $\mathrm{Cl}^C \simeq \mathrm{Pic}$ for both S and $S \times \mathbf{A}^1$, so by considering the sheaves $\mathcal{D}iv^C$, we reduce to the case where $S = \mathrm{Spec}(A)$ is affine and A is an integrally closed domain. We identify $S \times \mathbf{A}^1$ with $\mathrm{Spec}(A[X])$.

Take a line bundle $I \in \mathrm{Pic}(A[X])$. Since $K[X]$ is a UFD, we have $\mathrm{Pic}(K[X])$ is trivial by Proposition 3.3.4. Therefore $I \otimes_{A[X]} K[X] \simeq K[X]$, so we can assume that I is an ideal in $A[X]$. Divide by X until $I \cap A \neq 0$. We want to show that $I = (I \cap A)[X]$. Proposition 2.1.6 implies that

$$I = \bigcap_{\mathrm{ht}(\mathfrak{P})=1} I_{\mathfrak{P}} \subset K(X).$$

Let $\mathfrak{P}$ be a height 1 prime in $A[X]$ and set $\mathfrak{p} = \mathfrak{P} \cap A$, which is a prime in A of height ≤ 1. Then $A_{\mathfrak{p}}[X] \subset A[X]_{\mathfrak{P}}$ and $I_{\mathfrak{p}} = IA_{\mathfrak{p}}[X] = I_{\mathfrak{P}} \cap A_{\mathfrak{p}}[X]$. Since A satisfies (R_1), we know that $A_{\mathfrak{p}}$ is a DVR and hence a UFD. Therefore $I_{\mathfrak{p}} \in \mathrm{Pic}(A_{\mathfrak{p}}[X]) = 0$ is a principal ideal. Its generator must lie in $A_{\mathfrak{p}}$ since $I \cap A \neq 0$. It follows that $I_{\mathfrak{p}} = (I_{\mathfrak{P}} \cap A_{\mathfrak{p}})[X]$. Therefore

$$I = \bigcap_{\mathrm{ht}(\mathfrak{P})=1} (I_{\mathfrak{P}} \cap A_{\mathfrak{p}})[X] = (I \cap A)[X]. \qquad \square$$

Theorem 3.3.9 (Auslander-Buchsbaum)**.** *A regular local ring is a UFD.*

Proof. We suggest reading the proof in [GD67, Théorème 21.11.1] due to Kaplansky. Alternatively, see [Mat80, (19.B) Theorem 48] or [Mat89, Theorem 20.3] for a slightly different argument. □

4. Conclusion

4.1. **Further reading.** These notes are by no means a comprehensive review of local algebra – they only skim the surface of what is there. To name a few topics omitted: complete intersection rings, Gorenstein rings, and how flatness relates to local properties. See [Mat80] or [Mat89] for results in these areas. There are then of course even deeper topics that I know nothing about, which can be found in *loc. cit.* and EGA.

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