Geometric Periods

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Relative Aspects of the Langlands Program, L-Functions and Beyond Endoscopy
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- F = global field, $G = \mathrm{GL}_2$, $H = \mathbb{G}_m = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$
- $[G] = G(F)\backslash G(\mathbb{A})$

Theorem (Hecke, Maaß)

- *Let* $f \in \mathcal{A}_{\mathrm{cusp}}([G]/K)$ *be an unramified eigenform.*
- *Assume* f *Whittaker normalized.*

Then

$$L(\tfrac{1}{2}, \pi_f, \mathbf{std}) = \int_{[H]} f$$

Relative Langlands duality (Ben-Zvi–Sakellaridis–Venkatesh)

This is an instance of duality between Hamiltonian varieties

$$(T^*\check{X} \circlearrowleft \check{G}) \longleftrightarrow (T^*X \circlearrowleft G)$$

$$\check{X} = \mathbf{std} = \mathbb{A}^2 \longleftrightarrow X = H\backslash G = \mathbb{G}_m\backslash \mathrm{GL}_2$$

(In this case X and $\check{X}$ are both spherical.)

BZSV[1] say that period formula should be understood as

$$\text{Spectral } \check{X}\text{-period} = \text{Automorphic } X\text{-period}$$

(not just on cuspidal spectrum!)

- $f : [G] \to \mathbb{C}$ automorphic form
- $\sigma_f : \Gamma_F \to \check{G}$ the parameter of f

$$\text{Spectral period} \sum_{x \in \check{X} \text{ fixed by } \sigma_f} L(\mathcal{T}_x) = \text{Automorphic period} \int_{[H]} f$$

- The sum runs over *derived* fixed points of $\check{X}$ (when they are isolated) and $L(\mathcal{T}_x)$ is the L-value of the tangent complex $\mathcal{T}_x$.

[1] Ben-Zvi–Sakellaridis–Venkatesh

To understand derived nature of spectral period (among other reasons), we want to geometrize/categorify periods via geometric Langlands.

- $F = \mathbb{F}_q(C)$, $C_{/\mathbb{F}_q}$ smooth projective curve, $G = \mathrm{GL}_2$

Automorphic side

$$\begin{array}{ccc} C^\infty_{\mathrm{cusp}}([G]/G(\mathbb{O})) & \xrightarrow{\int_{[H]} f} & \mathbb{C} \\ \text{functions–sheaves} \rightsquigarrow & & \rightsquigarrow \\ \mathrm{Shv}_{\mathrm{cusp}}(\mathrm{Bun}_G) & \xrightarrow{\mathcal{P}_X} & \mathrm{Vect} \end{array}$$

- Everything is derived, e.g., $\mathrm{Vect} = D(\mathrm{Vect})$.
- While $\int_{[H]} f$ only converges sometimes / needs to be regularized, *the functor $\mathcal{P}_X$ does not need regularization since we can have ∞-vector spaces*.

Spectral side

- $f \longleftrightarrow \sigma_f : \pi_1(C) \to \check{G}$
- $\sigma \longleftrightarrow E_\sigma$ rank 2 local system on C
- $L(s, \pi, \mathbf{std}) = L(s, \sigma)$ Langlands L-function: given by Frobenius eigenvalues
- Grothendieck–Lefschetz trace formula:

$$L(s, \sigma) = \mathrm{Tr}(\mathrm{Frob}_q q^{-s}, \mathrm{Sym}\, H^\bullet_{\text{ét}}(C_{\overline{\mathbb{F}}_q}, E_\sigma))$$

$\mathbb{C}_{E_\sigma} \mapsto \mathrm{Sym}\, H^\bullet_{\text{ét}}(C_{\overline{\mathbb{F}}_q}, E_\sigma)$ $\mathcal{P}_X : D(\mathrm{Bun}_G) \to \mathrm{Vect}$

Categorical (de Rham) geometric Langlands conjecture (GLC)

Replace $\overline{\mathbb{F}}_q$ with $\mathbb{C}$, sheaves with complexes of D-modules.

$$\mathrm{IndCoh}_{\mathcal{N}ilp}(\mathrm{LocSys}_{\check{G}}) \cong D(\mathrm{Bun}_G)$$

$$\text{skyscraper } \mathbb{C}_E \longleftrightarrow \text{Hecke eigensheaf } \mathcal{F}_E$$

- $\mathrm{LocSys}_{\check{G}}$ is the stack of rank 2 local systems on C

Want: Functors $\mathrm{IndCoh}_{\mathcal{N}ilp}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{Vect} \overset{\mathrm{GLC}}{\longleftrightarrow} D(\mathrm{Bun}_G) \to \mathrm{Vect}$

- By miraculous duality [Drinfeld–Gaitsgory], equivalent to asking for specific **objects** to match in $\mathrm{IndCoh}_{\mathcal{N}ilp}(\mathrm{LocSys}_{\check{G}}) \cong D(\mathrm{Bun}_G)$.

Relative Langlands Duality: $\check{X} = \mathbf{std}$, $X = \mathbb{G}_m \backslash \mathrm{GL}_2$

Spectral period $\in \mathrm{IndCoh}_{\mathcal{N}ilp}(\mathrm{LocSys}_{\check{G}})$

$$\mathrm{LocSys}_{\check{G}}^{\check{X}} := \mathrm{Maps}(C_{\mathrm{dR}}, \check{X}/\check{G})$$

$$\downarrow \Pi^{\mathrm{spec}}$$

$$\mathrm{LocSys}_{\check{G}}$$

$\mathrm{LocSys}_{\check{G}}^{\check{X}}$ is a *derived* stack:
$(\Pi^{\mathrm{spec}})^{-1}(\{E\}) = R\Gamma_{\mathrm{dR}}(C, E)$

Automorphic period $\in D(\mathrm{Bun}_G)$

$$\mathrm{Bun}_G^X := \mathrm{Maps}(C, X/G)$$

$$\downarrow \Pi$$

$$\mathrm{Bun}_G$$

$\mathrm{Bun}_G^X = \mathrm{Bun}_H \to \mathrm{Bun}_G$
is analog of $[H] \to [G]$

Conjecture (Drinfeld, Ben-Zvi–Sakellaridis–Venkatesh)

$$(\Pi^{\mathrm{spec}})_*^{\mathrm{IndCoh}}(\omega_{\mathrm{LocSys}_{\check{G}}^{\check{X}}}) \overset{\mathrm{GLC}}{\longleftrightarrow} \Pi_!(\mathbb{C}_{\mathrm{Bun}_G^X})$$

Toy model: $R\Gamma(V, \mathcal{O}_V) = \mathrm{Sym}\, V^*$. Taking $V = R\Gamma_{\mathrm{dR}}(C, E)$ "almost" recovers L-function.

$\check{X} = \mathbf{std},\ X = \mathbb{G}_m \backslash \mathrm{GL}_2$

$$\begin{array}{ccc} R\Gamma_{\mathrm{dR}}(C, E) & \longrightarrow & \mathrm{LocSys}_{\check{G}}^{\check{X}} \\ \downarrow & & \downarrow \Pi^{\mathrm{spec}} \\ \mathrm{pt} & \xrightarrow{E} & \mathrm{LocSys}_{\check{G}} \end{array} \qquad\qquad \begin{array}{c} \mathrm{Bun}_G^X := \mathrm{Bun}_H \\ \downarrow \Pi \\ \mathrm{Bun}_G \end{array}$$

- $\check{X} = \mathbb{A}^2$ has two $\check{G}$-orbits: 0 and $\mathbb{A}^2 - 0 = \check{G}/\check{\mathrm{M}}\mathrm{ir}_2$
- This gives a distinguished triangle in local/coherent cohomology

$$\mathscr{L}_{\mathrm{std}} \to \Pi_*^{\mathrm{spec}}(\omega_{\mathrm{LocSys}_{\check{G}}^{\check{X}}}) \to \mathrm{Eis}_{\check{\mathrm{M}}\mathrm{ir}_2}^{\mathrm{spec}}(\omega_{\mathrm{LocSys}_{\mathbb{G}_m}}) \to [1]$$

where !-stalk of $\mathscr{L}_{\mathrm{std}}$ at E is $\mathrm{Sym}\, H^{\bullet}_{\mathrm{dR}}(C, E)$.

Theorem (Feng-W)

There exist distinguished triangles with "graded pieces" matching under GLC: `(Shifts omitted.)`

$$
\begin{array}{ccccccc}
\mathscr{L}_{\mathrm{std}} & \longrightarrow & \Pi_*^{\mathrm{spec}}(\omega_{\mathrm{LocSys}_{\check{G}}^{\check{X}}}) & \longrightarrow & \mathrm{Eis}^{\mathrm{spec}}_{\check{\mathrm{Mir}}_2}(\omega_{\mathrm{LocSys}_{\mathbb{G}_m}}) & \longrightarrow & [1] \\
\updownarrow \text{GLC} & & & & \updownarrow \text{GLC} & & \\
\mathbb{T}_{\mathrm{Sym}(\mathbf{std})}(\mathrm{Whit}) & \longrightarrow & \Pi_!(\mathbb{C}_{\mathrm{Bun}_G^X}) & \longrightarrow & \mathrm{Eis}_{\mathrm{Mir}_2,!}(\mathbb{C}_{\mathrm{Bun}_{\mathbb{G}_m}}) & \longrightarrow & [1]
\end{array}
$$

Proof: Unfolding.