Geometric Hecke Periods

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- F = global field, $G = \mathrm{GL}_2$, $H = \mathbb{G}_m = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$
- $[G] = G(F)\backslash G(\mathbb{A})$

Theorem (Hecke, Maaß)

- *Let $f \in \mathcal{A}_{\mathrm{cusp}}([G]/K)$ be an unramified eigenform.*
- *Assume f Whittaker normalized.*

Then

$$L(\tfrac{1}{2}, \pi_f, \mathbf{std}) = \int_{[H]} f$$

Relative Langlands duality (Ben-Zvi–Sakellaridis–Venkatesh)

This is an instance of duality between Hamiltonian varieties

$$(T^*\check{X} \circlearrowleft \check{G}) \longleftrightarrow (T^*X \circlearrowleft G)$$
$$\check{X} = \mathbf{std} = \mathbb{A}^2 \longleftrightarrow X = H\backslash G = \mathbb{G}_m\backslash \mathrm{GL}_2$$

Question: Can we categorify Hecke period via geometric Langlands?

- $F = \mathbb{F}_q(C)$, $C_{/\mathbb{F}_q}$ smooth projective curve, $G = \mathrm{GL}_2$

Automorphic side

$$\begin{tikzcd} C^\infty_{\mathrm{cusp}}([G]/G(\mathbb{O})) \arrow[r, "\int_{[H]} f"] \arrow[d, squiggly, "\text{functions–sheaves}"'] & \mathbb{C} \arrow[d, squiggly] \\ \mathrm{Shv}_{\mathrm{cusp}}(\mathrm{Bun}_G) \arrow[r, "\mathcal{P}_X"] & \mathrm{Vect} \end{tikzcd}$$

Spectral side

- Langlands correspondence: $\pi_f \longleftrightarrow \sigma : \pi_1(C) \to \check{G}$
- $\sigma \longleftrightarrow E_\sigma$ rank 2 local system on C
- $L(s, \pi, \mathbf{std}) = L(s, \sigma)$ Artin L-function: given by Frobenius eigenvalues
- Grothendieck–Lefschetz trace formula:

$$L(s,\sigma) = \mathrm{Tr}(\mathrm{Frob}_q q^{-s}, \mathrm{Sym}\, H^\bullet_{\text{ét}}(C_{\overline{\mathbb{F}}_q}, E_\sigma))$$

$\mathbb{C}_{E_\sigma} \mapsto \mathrm{Sym}\, H^\bullet_{\text{ét}}(C_{\overline{\mathbb{F}}_q}, E_\sigma)$ $\mathcal{P}_X : D(\mathrm{Bun}_G) \to \mathrm{Vect}$

Categorical geometric Langlands conjecture (GLC)

Replace $\overline{\mathbb{F}}_q$ with $\mathbb{C}$, sheaves with complexes of D-modules.

$$\mathrm{QCoh}'(\mathrm{Loc}_{\check{G}}) \cong D(\mathrm{Bun}_G)$$

$$\text{skyscraper } \mathbb{C}_E \longleftrightarrow \text{Hecke eigensheaf } \mathcal{F}_E$$

- $\mathrm{Loc}_{\check{G}}$ is the stack of rank 2 local systems on C

Want: Functors $\mathrm{QCoh}'(\mathrm{Loc}_{\check{G}}) \to \mathrm{Vect} \overset{\mathrm{GLC}}{\longleftrightarrow} D(\mathrm{Bun}_G) \to \mathrm{Vect}$

- By miraculous duality [Drinfeld–Gaitsgory], equivalent to asking for specific **objects** to match in $\mathrm{QCoh}'(\mathrm{Loc}_{\check{G}}) \cong D(\mathrm{Bun}_G)$.

Relative Langlands Duality: $\check{X} = \mathbf{std}$, $X = \mathbb{G}_m \backslash \mathrm{GL}_2$

$\mathbb{C}_{E_\sigma} \mapsto \mathrm{Sym}\, H^\bullet_{\text{ét}}(C_{\overline{\mathbb{F}}_q}, E_\sigma)$

$\mathcal{P}_X : D(\mathrm{Bun}_G) \to \mathrm{Vect}$

$$\begin{tikzcd} R\Gamma_{\mathrm{dR}}(C,E) \arrow[r] \arrow[d] & \mathrm{Loc}_{\check{X}} \arrow[d, "\Pi^{\mathrm{spec}}"'] \\ \mathrm{pt} \arrow[r, "E"] & \mathrm{Loc}_{\check{G}} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Bun}_X := \mathrm{Bun}_H \arrow[d, "\Pi"] \\ \mathrm{Bun}_G \end{tikzcd}$$

Conjecture (Drinfeld, Ben-Zvi–Sakellaridis–Venkatesh)

$$\Pi^{\mathrm{spec}}_*(\omega_{\mathrm{Loc}_{\check{X}}}) \overset{\mathrm{GLC}}{\longleftrightarrow} \Pi_!(\mathbb{C}_{\mathrm{Bun}_X})$$

$$\mathrm{QCoh}'(\mathrm{Loc}_{\check{G}}) \overset{\mathrm{GLC}}{\cong} D(\mathrm{Bun}_G)$$

$$\begin{tikzcd} R\Gamma_{\mathrm{dR}}(C, E) \arrow[r] \arrow[d] & \mathrm{Loc}_{\check{X}} \arrow[d, "\Pi^{\mathrm{spec}}"'] \\ \mathrm{pt} \arrow[r, "E"] & \mathrm{Loc}_{\check{G}} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Bun}_X := \mathrm{Bun}_H \arrow[d, "\Pi"] \\ \mathrm{Bun}_G \end{tikzcd}$$

- $\check{X} = \mathbb{A}^2$ has two $\check{G}$-orbits: 0 and $\mathbb{A}^2 - 0 = \check{G}/\check{\mathrm{M}}\mathrm{ir}_2$.
- This gives a distinguished triangle

$$\mathscr{L}_{\mathrm{std}} \to \Pi^{\mathrm{spec}}_*(\omega_{\mathrm{Loc}_{\check{X}}}) \to \mathrm{Eis}^{\mathrm{spec}}_{\check{\mathrm{M}}\mathrm{ir}_2}(\omega_{\mathrm{Loc}_{\mathbb{G}_m}}) \to [1]$$

Theorem (Feng-W)

There exist distinguished triangles with "graded pieces" matching under GLC:

$$\begin{tikzcd} \mathscr{L}_{\mathrm{std}} \arrow[r] \arrow[d, leftrightarrow, "\mathrm{GLC}"] & \Pi^{\mathrm{spec}}_*(\omega_{\mathrm{Loc}_{\check{X}}}) \arrow[r] & \mathrm{Eis}^{\mathrm{spec}}_{\mathrm{Mir}_2}(\omega_{\mathrm{Loc}_{\mathbb{G}_m}}) \arrow[r] \arrow[d, leftrightarrow, "\mathrm{GLC}"] & {[1]} \\ \mathbb{T}_{\mathrm{Sym}(\mathbf{std})}(\mathrm{Whit}) \arrow[r] & \Pi_!(\mathbb{C}_{\mathrm{Bun}_X}) \arrow[r] & \mathrm{Eis}_{\mathrm{Mir}_2,!}(\mathbb{C}_{\mathrm{Bun}_{\mathbb{G}_m}}) \arrow[r] & {[1]} \end{tikzcd}$$