Derived Satake equivalence for Godement–Jacquet monoids

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Outline

The main equivalence of categories

- Let $k = \overline{\mathbb{F}}_q$ or $\mathbb{C}$.
- $F = k(\!(t)\!)$ and $\mathcal{O} = k[\![t]\!]$.

Theorem (Tsao-Hsien Chen–W)

There exists an equivalence of categories

$$D_c^!(\mathrm{GL}_n(\mathcal{O})\backslash \mathrm{M}_n(F)/\mathrm{GL}_n(\mathcal{O})) \cong \mathrm{Perf}(\mathrm{GL}_n \times \mathfrak{gl}_n^*[2] \times \mathrm{V}[2] \times \mathrm{V}^*)^{\mathrm{GL}_n \times \mathrm{GL}_n}$$

where

- M_n *is monoid of* $n \times n$*-matrices,*
- *RHS means perfect complexes of* $\mathrm{GL}_n \times \mathrm{GL}_n$*-equivariant* $k[\mathrm{GL}_n] \otimes \mathrm{Sym}(\mathfrak{gl}_n[-2] \times \mathrm{V}^*[-2] \times \mathrm{V})$*-modules.*

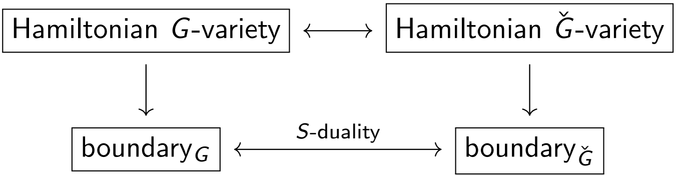
Note: $\mathrm{GL}_n \times \mathfrak{gl}_n^* \times \mathrm{V} \times \mathrm{V}^* = T^*(\mathrm{GL}_n \times \mathrm{V})$

Connections to physics

- $G = \mathrm{GL}_n \times \mathrm{GL}_n = \check{G}$

$$\boxed{\mathrm{M}_n T^*\mathrm{M}_n \circlearrowleft G} \rightsquigarrow \boxed{T^*(\mathrm{GL}_n \times \mathrm{V}) \circlearrowleft \check{G}}$$

- $D_c^!(\mathrm{M}_n(F))$ is quantization of $T^*\mathrm{M}_n(F)$.
- $T^*\mathrm{M}_n \longleftrightarrow T^*(\mathrm{GL}_n \times \mathrm{V})$ is a special case of duality between Hamiltonian varieties.
- S-duality in super Yang–Mills $\mathcal{N} = 4$ $d = 4$ TQFT[1] matches boundary theories
- Gaiotto–Witten: Hamiltonian G-variety $\rightsquigarrow$ boundary theory



[1] Kapustin–Witten: this is geometric Langlands TQFT

$$\boxed{T^*\mathrm{M}_n \circlearrowleft G} \longleftrightarrow \boxed{T^*(\mathrm{GL}_n \times \mathrm{V}) \circlearrowleft \check{G}}$$

Taking $\mathcal{A}$-twist on left and $\mathcal{B}$-twist on right of boundary theories predicts our equivalence:

$$D_c^!(\mathrm{M}_n(F)/G(\mathcal{O})) = \mathrm{Perf}(T^*(\mathrm{GL}_n \times V)[?])^{\check{G}}$$

Can swap! $\mathcal{B}$-twist on left and $\mathcal{A}$-twist on right:

Theorem (Braverman–Finkelberg–Ginzburg–Travkin)

There is equivalence of categories

$$\mathrm{Perf}(\mathrm{M}_n[2] \times \mathrm{M}_n^*)^{\mathrm{GL}_n \times \mathrm{GL}_n} \cong D_c^!((\mathrm{GL}_n(\mathcal{O}) \backslash \mathrm{GL}_n(F) \times \mathrm{V}(F))/\mathrm{GL}_n(\mathcal{O}))$$

Connections to number theory

Connection is due to forthcoming work of Ben-Zvi–Sakellaridis–Venkatesh.

$$\boxed{T^*\mathrm{M}_n \circlearrowleft G} \longleftrightarrow \boxed{T^*(\mathrm{GL}_n \times \mathrm{V}) \circlearrowleft \check{G}}$$

- Rankin–Selberg convolution: some integral involving $\mathrm{GL}_n \times V$ produces L-function for representation $\mathrm{V} \otimes \mathrm{V} \in \mathrm{Rep}(\mathrm{GL}_n \times \mathrm{GL}_n)$.
- Godement–Jacquet: some integral involving M_n produces L-function for $\mathrm{V} \in \mathrm{Rep}(\mathrm{GL}_n)$.
- "Induction" of $T^*\mathrm{V}$ from $\Delta\mathrm{GL}_n$ to $\mathrm{GL}_n \times \mathrm{GL}_n$ gives $T^*(\mathrm{GL}_n \times V)$.

Derived Satake equivalence

Theorem (Bezrukavnikov–Finkelberg, after Ginzburg)

For any reductive group G, there is equivalence of monoidal categories

$$\mathrm{Sph}_G^{loc.c} := D_c(G(\mathcal{O})\backslash G(F)/G(\mathcal{O})) \cong \mathrm{Perf}(\check{\mathfrak{g}}^*[2])^{\check{G}}$$

Theorem (Tsao-Hsien Chen–W)

Let $G = \mathrm{GL}_n \times \mathrm{GL}_n$. The equivalence

$$D_c^!(\mathrm{M}_n(F)/G(\mathcal{O})) \cong \mathrm{Perf}(\mathrm{GL}_n \times \mathfrak{gl}_n^*[2] \times \mathrm{V}[2] \times \mathrm{V}^*)^{\check{G}}$$

is compatible with action of $\mathrm{Sph}_G^{loc.c}$, where

- $\mathrm{Sph}_G^{loc.c}$ *acts on left by convolution.*
- $\mathrm{Perf}(\check{\mathfrak{g}}^*[2])^{\check{G}}$ *acts on right by pullback under moment map*

$$T^*(\mathrm{GL}_n \times \mathrm{V}) \to \check{\mathfrak{g}}^*.$$

Categories of sheaves

- In derived Satake, $\mathrm{GL}_n(F)/\mathrm{GL}_n(\mathcal{O}) = \mathrm{Gr}_{\mathrm{GL}_n}$ is ind-finite type.
- However, $\mathrm{M}_n(F)/\mathrm{GL}_n(\mathcal{O})$ is infinite type, so we need constructible sheaf theory on infinite type stacks.

Two good sheaf theories $D^!$ and D_*:

- $\mathrm{M}_n(F) = \mathrm{colim}_{i\in\mathbb{N}}\, t^{-i}\mathrm{M}_n(\mathcal{O}) = \mathrm{colim}_{i\in\mathbb{N}} \lim_{j\in\mathbb{N}} t^{-i}\mathrm{M}_n(\mathcal{O})/t^j\mathrm{M}_n(\mathcal{O})$
- $X_{ij} := t^{-i}\mathrm{M}_n(\mathcal{O})/t^j\mathrm{M}_n(\mathcal{O})$ is finite type scheme, and $G(\mathcal{O})$ acts through a finite type quotient $G(\mathcal{O}) \twoheadrightarrow G_{ij}$ with unipotent kernel.

Definition

$$D^!_c(\mathrm{M}_n(F)/G(\mathcal{O})) = \underset{i!}{\mathrm{colim}}\, \overset{!}{\underset{j}{\mathrm{colim}}}\, D_c(X_{ij}/G_{ij})$$

$$D_{*c}(\mathrm{M}_n(F)/G(\mathcal{O})) = \underset{i*}{\mathrm{colim}}\, \overset{*}{\underset{j}{\mathrm{colim}}}\, D_c(X_{ij}/G_{ij})$$

Analogy:

$D^!_c \leftrightarrow$ compactly supported smooth functions

$D_{*c} \leftrightarrow$ compactly supported smooth measures

Because M_n is **smooth**, we have:

Theorem (Tsao-Hsien Chen–W)

We have commutative diagram of equivalences

$$\begin{array}{ccc} D^!_c(\mathrm{M}_n(F)/G(\mathcal{O})) & \xrightarrow[\sim]{\Phi^{2,0}} & \mathrm{Perf}(\mathfrak{gl}_n^*[2] \times \mathrm{V}[2] \times \mathrm{V}^*)^{\mathrm{GL}_n} \\ {\scriptstyle shift}\downarrow\wr & & \wr\downarrow{\scriptstyle shift\ in\ coh.} \\ D_{*c}(\mathrm{M}_n(F)/G(\mathcal{O})) & \xrightarrow[\sim]{\Phi^{0,2}} & \mathrm{Perf}(\mathfrak{gl}_n^*[2] \times \mathrm{V} \times \mathrm{V}^*[2])^{\mathrm{GL}_n} \end{array}$$

Right arrow does not come from an isomorphism of dg algebras.

Fourier transform

Since M_n can be thought of as a vector space, we can define Fourier transform on sheaves/functions of it.

Theorem (Tsao-Hsien Chen–W)

We have commutative diagram of equivalences

$$\begin{array}{ccc} D_{*c}(\mathrm{M}_n(F)/G(\mathcal{O})) & \xrightarrow[\sim]{\Phi^{0,2}} & \mathrm{Perf}(\mathfrak{gl}_n^*[2] \times \mathrm{V} \times \mathrm{V}^*[2])^{\mathrm{GL}_n} \\ \mathrm{FT}\downarrow\sim & & \sim\downarrow \mathit{swap} \\ D^!_c(\mathrm{M}_n(F)/G(\mathcal{O})) & \xrightarrow[\sim]{\Phi^{2,0}} & \mathrm{Perf}(\mathfrak{gl}_n^*[2] \times \mathrm{V}[2] \times \mathrm{V}^*)^{\mathrm{GL}_n} \end{array}$$

where swap *is pullback along* $\mathrm{V}[2] \times \mathrm{V}^* \to \mathrm{V} \times \mathrm{V}^*[2] : (v, \xi) \mapsto (\xi, v)$ *after identifying* $\mathrm{V} \cong \mathrm{V}^*$ *(up to Chevalley automorphism)*

We do not need the commutativity of this diagram to prove $\Phi^{2,0}$ is an equivalence!

Equivariant cohomology

- $f^!$ is naturally defined on $D^!_c$ and f_* naturally defined on D_{*c}.
- Consider the maps

$$\mathrm{pt} \xrightarrow{0} \mathrm{M}_n(F) \quad \text{and} \quad \mathrm{M}_n(F) \xrightarrow{p} \mathrm{pt}$$

  Both maps very not finite type.

Consider functors

$$D^!_c(\mathrm{M}_n(F)/G(\mathcal{O})) \xrightarrow{0^!} D_c(\mathrm{pt}/G(\mathcal{O})) = \mathrm{Perf}(H^\bullet(BG))$$
$$D_{*c}(\mathrm{M}_n(F)/G(\mathcal{O})) \xrightarrow{p_*} D_c(\mathrm{pt}/G(\mathcal{O})) = \mathrm{Perf}(H^\bullet(BG))$$

- $p_* = R\Gamma_{G(\mathcal{O})}(\mathrm{M}_n(F), -)$ is equivariant cohomology
- $0^! = R\Gamma_{G(\mathcal{O}),c}(\mathrm{M}_n(F), -)$ by contraction principle
- $H^\bullet_{G(\mathcal{O})}(\mathrm{M}_n(F), \mathbb{C}) = H^\bullet(BG) = \mathbb{C}[\mathfrak{t}[2]]^W$

Kostant–Weierstraß sections

What do $0^!$ and p_* correspond to spectrally?
Fact: $T^*(\mathrm{GL}_n \times \mathrm{V}) /\!\!/ \check{G} = \mathfrak{t} /\!\!/ W = \mathbb{A}^n \times \mathbb{A}^n$ (recall $\check{G} = \mathrm{GL}_n \times \mathrm{GL}_n$)
Quotient identifies with invariant moment map

$$T^*(\mathrm{GL}_n \times \mathrm{V}) \to \check{\mathfrak{g}}^* \to \check{\mathfrak{g}}^* /\!\!/ \check{G}.$$

The invariant moment map has **two** inequivalent sections

$$\kappa^{2,0} : \mathfrak{t} /\!\!/ W = \mathbb{A}^n \times \mathbb{A}^n \to \mathrm{GL}_n \times \mathfrak{gl}_n^* \times \mathrm{V} \times \mathrm{V}^*$$

$$(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}) \mapsto \mathrm{Id}, \begin{pmatrix} 0 & & & a_0 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & \vdots \\ & & 1 & a_{n-1} \end{pmatrix}, b - a, e_n^*$$

and $\kappa^{0,2}(a, b) = (\mathrm{Id}, \kappa(a)^T, e_n, (b - a)^T)$.

The $\check{G}$-action on $T^*(\mathrm{GL}_n \times \mathrm{V})$ extends $\kappa^{2,0}$ to open embedding

$$\tilde{\kappa}^{2,0} : \check{G} \times \mathfrak{t}/\!\!/W \hookrightarrow T^*(\mathrm{GL}_n \times \mathrm{V})$$

and similarly for $\kappa^{0,2} \rightsquigarrow \tilde{\kappa}^{0,2}$.

- Image of $\tilde{\kappa}^{2,0}$ is open with complement of codimension 1 and same for $\tilde{\kappa}^{0,2}$
- **Key Fact:** the union of two images has complement of codimension 2

$$\begin{array}{ccc} D^!_c(\mathrm{M}_n(F)/G(\mathcal{O})) & \xrightarrow{\quad 0^! \quad} & D_c(\mathrm{pt}/G(\mathcal{O})) \\ {\scriptstyle \Phi^{2,0}}\downarrow\wr & & \downarrow\wr \\ \mathrm{Perf}^{[?]}(T^*(\mathrm{GL}_n \times \mathrm{V}))^{\check{G}} & \xrightarrow{(\tilde{\kappa}^{2,0})^*} & \mathrm{Perf}^{[?]}(\check{G} \times \mathfrak{t}/\!\!/W)^{\check{G}} \end{array}$$

and similarly $p_* = H^\bullet_{G(\mathcal{O})}(\mathrm{M}_n(F), -)$ corresponds to $(\tilde{\kappa}^{0,2})^*$.

Proof sketch

Want: $\Phi^{2,0} : D^!_c(\mathrm{M}_n(F)/G(\mathcal{O})) \cong \mathrm{Perf}(\mathrm{GL}_n \times \mathfrak{gl}^*_n[2] \times \mathrm{V}[2] \times \mathrm{V}^*)^{\check{G}}$.

By (non-derived) geometric Satake, $\mathrm{Rep}(\check{G}) = \mathrm{Perv}(G(\mathcal{O})\backslash G(F)/G(\mathcal{O}))$ acts on $D^!_c(\mathrm{M}_n(F)/G(\mathcal{O}))$.

Fact: $\mathrm{Rep}(\check{G})$ action on $\omega_{\mathrm{M}_n(\mathcal{O})}$ generates $D^!_c(\mathrm{M}_n(F)/G(\mathcal{O}))$.

- Consider de-equivariantized algebra

$$A = R\,\mathrm{Hom}_{D^!_c}(\omega_{\mathrm{M}_n(\mathcal{O})}, \omega_{\mathrm{M}_n(\mathcal{O})} \star k[\check{G}]).$$

- Apply $0^!$ and $p_* \circ \mathrm{shift}$ to get maps $A \rightrightarrows k[\check{G} \times \mathfrak{t}/\!\!/W]$ (with shifts)
- By purity argument, A is formal and above maps are injective.
- Define map $\phi : k[\mathrm{GL}_n] \otimes \mathrm{Sym}(\mathfrak{gl}_n[-2] \times \mathrm{V}^*[-2] \times \mathrm{V}) \to A$ by explicit generators + derived Satake
- Check compositions equal $\tilde{\kappa}^{2,0}, \tilde{\kappa}^{0,2}$.
- Codimension 2 implies ϕ is isomorphism.

Thank you!