Derived Satake equivalence for Godement–Jacquet monoids

Jonathan Wang (joint w/ Tsao-Hsien Chen)

Perimeter Institute

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Outline

Overview

- $F = \mathbb{F}_q(\!(t)\!)$ and $\mathcal{O} = \mathbb{F}_q[\![t]\!]$.
- $\mathrm{M}_n = n \times n$ matrices
- Godement–Jacquet construct local standard L-function by spectrally decomposing $\mathcal{S}(\mathrm{M}_n(F))$
- Useful to study geometrization of $\mathcal{S}(\mathrm{M}_n(F))$.
- Functions–sheaves analogy: $\mathcal{S}(\mathrm{M}_n(F)) \rightsquigarrow D_c^!(\mathrm{M}_n(F))$
- Unramified situation: $D_c^!(\mathrm{GL}_n(\mathcal{O})\backslash \mathrm{M}_n(F)/\mathrm{GL}_n(\mathcal{O}))$
- Idea: Plancherel density $\|\Phi\|_\pi^2$ for $\Phi \in \mathcal{S}(\mathrm{M}_n(F))$ has to do with derived Hom

From now on, $F = \overline{\mathbb{F}}_q(\!(t)\!), \mathcal{O} = \overline{\mathbb{F}}_q[\![t]\!]$, $\mathsf{V} = \mathsf{std} \in \mathsf{Rep}(\mathsf{GL}_n)$.

Theorem (Tsao-Hsien Chen–W)

There exists an equivalence of categories

$$D_c^!(\mathsf{GL}_n(\mathcal{O})\backslash \mathsf{M}_n(F)/\mathsf{GL}_n(\mathcal{O})) \cong \mathsf{Perf}(\mathfrak{gl}_n^*[2] \times \mathsf{V}[2] \times \mathsf{V}^*)^{\mathsf{GL}_n}$$

where RHS means perfect complexes of GL_n*-equivariant* $\mathsf{Sym}(\mathfrak{gl}_n[-2] \times \mathsf{V}^*[-2] \times \mathsf{V})$*-modules.*

Note: $\mathsf{Perf}(\mathfrak{gl}_n^* \times \mathsf{V} \times \mathsf{V}^*)^{\mathsf{GL}_n} = \mathsf{Perf}(T^*(\mathsf{GL}_n \times \mathsf{V}))^{\mathsf{GL}_n \times \mathsf{GL}_n}$

Connection to Ben-Zvi–Sakellaridis–Venkatesh

- $G = \mathrm{GL}_n \times \mathrm{GL}_n = \check{G}$

$$\boxed{T^*\mathrm{M}_n \circlearrowleft G} \rightsquigarrow \boxed{T^*(\mathrm{GL}_n \times \mathrm{V}) \circlearrowleft \check{G}}$$

  is a special case of duality between affine Hamiltonian varieties.

- More generally, if X is a smooth affine spherical variety, then can define $\check{G}_X \subset \check{G}$ and $V_X \in \mathrm{Rep}(\check{G}_X)$.

Conjecture (BZSV)

Let X as above. Then $M^\vee := \check{G} \times^{\check{G}_X} V_X$ is Hamiltonian, and we have equivalence of categories

$$D_c^!(X(F)/G(\mathcal{O})) \cong \mathrm{Perf}^{[]}(M^\vee)^{\check{G}}$$

$T^*X \leftrightarrow M^\vee$

Godement–Jacquet:

$$\boxed{T^*\mathrm{M}_n \circlearrowleft G} \longleftrightarrow \boxed{T^*(\mathrm{GL}_n \times \mathrm{V}) \circlearrowleft \check{G}}$$

Can swap!

$$\boxed{T^*(\mathrm{GL}_n \times \mathrm{V}) \circlearrowleft G} \longleftrightarrow \boxed{T^*\mathrm{M}_n \circlearrowleft \check{G}}$$

Rankin–Selberg convolution: some integral involving

$$\mathrm{GL}_n \times V = \overline{\mathrm{Mir}\backslash \mathrm{GL}_n \times \mathrm{GL}_n}$$

produces L-function for representation $\mathrm{V} \otimes \mathrm{V} \in \mathrm{Rep}(\mathrm{GL}_n \times \mathrm{GL}_n)$.

Connection to Braverman–Kazhdan

- Let "det" : $G \to \mathbb{G}_m$ and $\rho \in \mathrm{Rep}(\check{G})$.
- Can combinatorially define an algebraic monoid M_ρ with group of units G
- $\mathrm{Perv}_{G(\mathcal{O})}(M_\rho(F)^\bullet)$ gives $\mathcal{S}_\rho(G(F))^{G(\mathcal{O})\times G(\mathcal{O})} \subset C^\infty(G(F))$ by Frobenius trace

Conjecture (Braverman–Kazhdan)

There exists $\mathcal{S}_\rho(G(F))$ such that zeta integrals give $L(s, \pi, \rho)$.

Conjecture

There is equivalence of abelian categories

$$\mathrm{Perv}_{G(\mathcal{O})}(M_\rho(F)^\bullet) \cong \wedge(\rho \oplus \rho^*)\text{-mod}_{\mathrm{fd}}^{\check{G}}$$

Derived Satake equivalence

Theorem (Bezrukavnikov–Finkelberg, after Ginzburg)

For any reductive group G, there is equivalence of monoidal categories

$$\mathrm{Sph}_G^{loc.c} := D_c(G(\mathcal{O})\backslash G(F)/G(\mathcal{O})) \cong \mathrm{Perf}(\check{\mathfrak{g}}^*[2])^{\check{G}}$$

Theorem (Tsao-Hsien Chen–W)

Let $G = \mathrm{GL}_n \times \mathrm{GL}_n$. The equivalence

$$D_c^!(\mathrm{M}_n(F)/G(\mathcal{O})) \cong \mathrm{Perf}(\mathrm{GL}_n \times \mathfrak{gl}_n^*[2] \times \mathrm{V}[2] \times \mathrm{V}^*)^{\check{G}}$$

is compatible with action of $\mathrm{Sph}_G^{loc.c}$, where

- $\mathrm{Sph}_G^{loc.c}$ *acts on left by convolution.*
- $\mathrm{Perf}(\check{\mathfrak{g}}^*[2])^{\check{G}}$ *acts on right by pullback under moment map*

$$T^*(\mathrm{GL}_n \times \mathrm{V}) \to \check{\mathfrak{g}}^*.$$

Fourier transform

Since M_n can be thought of as a vector space, we can define Fourier transform on sheaves/functions of it.

Theorem (Tsao-Hsien Chen–W)

We have commutative diagram of equivalences

$$\begin{array}{ccc} D_{*c}(\mathsf{M}_n(F)/G(\mathcal{O})) & \xrightarrow[\sim]{\Phi^{0,2}} & \mathrm{Perf}(\mathfrak{gl}_n^*[2] \times \mathsf{V} \times \mathsf{V}^*[2])^{\mathsf{GL}_n} \\ {\scriptstyle \mathsf{FT}}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle swap} \\ D_c^!(\mathsf{M}_n(F)/G(\mathcal{O})) & \xrightarrow[\sim]{\Phi^{2,0}} & \mathrm{Perf}(\mathfrak{gl}_n^*[2] \times \mathsf{V}[2] \times \mathsf{V}^*)^{\mathsf{GL}_n} \end{array}$$

where swap *is pullback along* $\mathsf{V}[2] \times \mathsf{V}^* \to \mathsf{V} \times \mathsf{V}^*[2] : (v, \xi) \mapsto (\xi, v)$ *after identifying* $\mathsf{V} \cong \mathsf{V}^*$ *(up to Chevalley automorphism)*

We do not need the commutativity of this diagram to prove $\Phi^{2,0}$ is an equivalence!

Equivariant cohomology

Can define functors

$$R\Gamma_{G(\mathcal{O}),c}(\mathrm{M}_n(F), -) : D^!_c(\mathrm{M}_n(F)/G(\mathcal{O})) \longrightarrow \mathrm{Perf}(H^\bullet(BG))$$
$$R\Gamma_{G(\mathcal{O})}(\mathrm{M}_n(F), -) : D_{*c}(\mathrm{M}_n(F)/G(\mathcal{O})) \longrightarrow \mathrm{Perf}(H^\bullet(BG))$$

$H^\bullet_{G(\mathcal{O})}(\mathrm{M}_n(F), \mathbb{C}) = H^\bullet(BG) = \mathbb{C}[\mathfrak{t}[2]]^W$

Kostant–Weierstraß sections

What do cohomology functors correspond to spectrally?

Fact: $T^*(\mathrm{GL}_n \times \mathrm{V})/\!\!/\check{G} = \mathfrak{t}/\!\!/W = \mathbb{A}^n \times \mathbb{A}^n$ (recall $\check{G} = \mathrm{GL}_n \times \mathrm{GL}_n$)

Quotient identifies with invariant moment map

$$T^*(\mathrm{GL}_n \times \mathrm{V}) \to \check{\mathfrak{g}}^* \to \check{\mathfrak{g}}^*/\!\!/\check{G}.$$

The invariant moment map has **two** inequivalent sections

$$\kappa^{2,0} : \mathfrak{t}/\!\!/W = \mathbb{A}^n \times \mathbb{A}^n \to \mathrm{GL}_n \times \mathfrak{gl}_n^* \times \mathrm{V} \times \mathrm{V}^*$$

$$(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}) \mapsto \mathrm{Id}, \begin{pmatrix} 0 & & & a_0 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & \vdots \\ & & 1 & a_{n-1} \end{pmatrix}, b - a, e_n^*$$

and $\kappa^{0,2}(a, b) = (\mathrm{Id}, \kappa(a)^T, e_n, (b-a)^T)$.

The $\check{G}$-action on $T^*(\mathrm{GL}_n \times \mathrm{V})$ extends $\kappa^{2,0}$ to open embedding

$$\tilde{\kappa}^{2,0} : \check{G} \times \mathfrak{t}/\!\!/W \hookrightarrow T^*(\mathrm{GL}_n \times \mathrm{V})$$

and similarly for $\kappa^{0,2} \rightsquigarrow \tilde{\kappa}^{0,2}$.

- Image of $\tilde{\kappa}^{2,0}$ is open with complement of codimension 1 and same for $\tilde{\kappa}^{0,2}$
- **Key Fact:** the union of two images has complement of codimension 2

$$\begin{array}{ccc} D_c^!(\mathrm{M}_n(F)/G(\mathcal{O})) & \xrightarrow{H^\bullet_{G(\mathcal{O}),c}} & \mathrm{Perf}(H^\bullet(BG)) \\ \Phi^{2,0}\downarrow\wr & & \downarrow\wr \\ \mathrm{Perf}^{[?]}(T^*(\mathrm{GL}_n \times \mathrm{V}))^{\check{G}} & \xrightarrow{(\tilde{\kappa}^{2,0})^*} & \mathrm{Perf}^{[?]}(\check{G} \times \mathfrak{t}/\!\!/W)^{\check{G}} \end{array}$$

and similarly $H^\bullet_{G(\mathcal{O})}(\mathrm{M}_n(F), -)$ corresponds to $(\tilde{\kappa}^{0,2})^*$.

Thank you!