FLAG VARIETY

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In this note, we explore some properties concerning the flag variety G/B where G is a reductive algebraic group and B a Borel subgroup over an algebraically closed field. We start with some very general facts, and make more assumptions to prove more interesting results later on.

1. G-equivariant sheaves and stacks

1.1. QCoh **on a stack.** We know that QCoh forms a stack, i.e., sheaf of groupoids, over $\mathbf{Sch}_{fpqc}(S)$ for any scheme S. Thus if we have an fpqc sheaf of groupoids $\mathcal{X}$ over S, we can define $\mathrm{QCoh}(\mathcal{X})$ as maps of sheaves

$$\mathcal{X} \to \mathrm{QCoh}$$

on $\mathbf{Sch}_{fpqc}(S)$. By 2-Yoneda, this definition agrees with the usual notion of quasi-coherent sheaves if $\mathcal{X}$ is a scheme.

For various other (usually equivalent) definitions in the case $\mathcal{X}$ is a DM/Artin stack, see [LMB00, Ch. 12-13].

1.2. **A different perspective on G-equivariant sheaves.** Let X be a k-scheme with [left] G-action, where G is affine algebraic group over algebraically closed field k. Let $\mathrm{QCoh}(X)^G$ denote G-equivariant sheaves on X.

Lemma 1. *There is an equivalence* $\mathrm{QCoh}(X)^G \simeq \mathrm{QCoh}([G\backslash X])$ *where* $[G\backslash X]$ *is the quotient stack.*

Proof. Start with $\mathcal{F} \in \mathrm{QCoh}(X)^G$. For $S \to [G\backslash X]$ corresponding to G-bundle $\mathcal{P}$, we have Cartesian square

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{\mathcal{P}} & [G\backslash X] \end{array}$$

Then $f^*\mathcal{F} \in \mathrm{QCoh}(\mathcal{P})^G$. We know that $\mathcal{P} \times_S \mathcal{P} \simeq G \times \mathcal{P}$, so G-equivariant structure on $f^*\mathcal{F}$ is the same as descent datum for $\mathcal{P} \to S$. This gives us a sheaf on S that pulls back to $f^*\mathcal{F}$. Next suppose we have $\mathcal{P}, \mathcal{P}' \in [G\backslash X](S)$ with isomorphism $\sigma : \mathcal{P}' \simeq \mathcal{P}$. Then there exists covering $S' \to S$ that trivializes both $\mathcal{P}, \mathcal{P}'$. So let us just consider

$$\begin{array}{ccccc} G \times S' & \xrightarrow{r_g} & G \times S' & \xrightarrow{f} & X \\ & & \downarrow & & \downarrow \\ & & S' & \longrightarrow & [G\backslash X] \end{array}$$

where $g \in G(S')$ and r_g denotes the corresponding G-equivariant map via right multiplication by g. Since $G \times S'$ is trivial, we have a section $i : S' \to G \times S'$

corresponding to 1. The two sheaves on S' are then just $i^*f^*\mathcal{F}, i^*r_g^*f^*\mathcal{F}$. Notice now that $g\cdot(f\circ i) = (f\circ r_g\circ i)\in X(S')$. Therefore G-equivariance of $\mathcal{F}$ gives an isomorphism

$$i^*f^*\mathcal{F}\simeq(f\circ i)^*\mathcal{F}\simeq(g\cdot(f\circ i))^*\mathcal{F}\simeq i^*r_g^*f^*\mathcal{F}.$$

By some kind of descent argument, this will give an isomorphism in the general case $\mathcal{P}\simeq\mathcal{P}'$ as well. This defines our map $[G\backslash X]\to \mathrm{QCoh}$.

For the other direction, just take $\mathcal{F}$ the image of the atlas $X\to[G\backslash X]\to\mathrm{QCoh}$. We get an isomorphism $\mathrm{act}^*\mathcal{F}\simeq p_2^*\mathcal{F}$ from the 2-Cartesian square

$$\begin{array}{ccc} G\times X & \xrightarrow{\mathrm{act}} & X \\ \downarrow{\scriptstyle p_2} & & \downarrow \\ X & \longrightarrow & [G\backslash X] \end{array}$$

The corresponding isomorphism $G\times G\times X\simeq G\times G\times X$ is given by $(g_1,g_2,x)\mapsto(g_1g_2,g_2,x)$ (I worked it out – I promise).

First, we show that if we start with $\mathcal{F}\in\mathrm{QCoh}(X)^G$, and we apply the above two functors, we essentially get back $\mathcal{F}$. First, $p_2^*\mathcal{F}\simeq\mathrm{act}^*\mathcal{F}$ tells us the quasi-coherent sheaf is $\mathcal{F}$. Next we check G-equivariant structure. Take the map $G\times X\to[G\backslash X]$ corresponding to

$$\begin{array}{ccc} G\times G\times X & \xrightarrow{g_1\cdot x} & X \\ \downarrow{\scriptstyle p_{23}} & & \\ G\times X & & \end{array}$$

Precompose with the $G\times G\times X\simeq G\times G\times X$ of the previous paragraph. Then precomposing with the identity section $G\times X\to G\times G\times X\rightrightarrows X$ gives p_2,act. The isomorphism $p_2^*\mathcal{F}\simeq\mathrm{act}^*\mathcal{F}$ thus gotten is exactly the original.

To check starting with a map $[G\backslash X]\to\mathrm{QCoh}$ should follow entirely from formal compatibility properties of stacks. □

Lemma 1 is true for left or right G-actions on X.

1.3. $\mathrm{QCoh}(BG)$**.** We have the algebraic stack $BG=[G\backslash\cdot]$. We characterize quasi-coherent sheaves on it.

Lemma 2. *There is an equivalence between* $\mathrm{QCoh}(BG)$ *and* G*-modules.*

Here by a G-module we mean what [MFK94, Ch. 1, §1] refers to as dual action of G on k-vector space V, or equivalently what [Jan03, I.2.7-8] calls an $\mathcal{O}_G$-comodule. This is the same as a rational representation when V is finite dimensional.

Proof. Start with a map $\mathcal{F}:BG\to\mathrm{QCoh}$. We have an atlas $\cdot\to BG$ corresponding to $G\to k$. Let V be $\mathcal{F}(\cdot\to BG)$. Now the isomorphism $\sigma:G\times G\simeq G\times G$ defined by m,p_2 must give an isomorphism $\mathcal{O}_G\otimes V\simeq\mathcal{O}_G\otimes V$. This gives a G-action on V.

We now show how $\mathcal{F}$ is determined by V. Take $S\xrightarrow{\mathcal{P}}BG$. Then we have some covering $S'\to S$ trivializing $\mathcal{P}$, i.e.,

$$S'\underset{S}{\times}S'\rightrightarrows S'\to S\to BG$$

such that $S' \to BG$ factors through $\cdot \to BG$. Let $S'' = S' \times_S S'$; the transition $G \times S'' \to G \times S''$ corresponds to some $g \in G(S'')$. This isomorphism corresponds to the pullback of σ along

$$S'' \overset{g(s),s}{\to} G \times S'' \overset{p_1}{\to} G.$$

We therefore have that $\mathcal{F}(S \overset{\mathcal{P}}{\to} BG)$ corresponds to the descent datum of $V \otimes \mathcal{O}_{S'}$ on S' with transitions the pullback of $\mathcal{F}(\sigma)$.

Now if we start with a representation V; that is, a map $V \to \mathcal{O}_G \otimes V$ satisfying certain conditions, we get a map $\mathcal{O}_G \otimes V \to \mathcal{O}_G \otimes V$, which is an isomorphism using the conditions. The previous paragraph in reverse tells us how to get a map $BG \to \mathrm{QCoh}$. □

Better proof. By Lemma 1, we have $\mathrm{QCoh}([G\backslash\cdot]) \simeq \mathrm{QCoh}(\cdot)^G$, where the latter is equivalent to $\mathcal{O}_G$-comodules (see 233A.9.1(a)). □

Lemma 2 is true for left or right G-bundles.

Corollary 3. *Let $H \subset G$ be a closed embedding of affine algebraic groups. There is an equivalence between $\mathrm{QCoh}(G/H)^G$ and H-modules.*

Proof. From Toly's talk, we have $[G\backslash(G/H)] \simeq [H\backslash\cdot] = BH$ (see Stacks.BG). Now Lemmas 1, 2 give the result.

As an exercise, let us explicitly describe the correspondences in both directions.

Start with $\mathcal{F} \in \mathrm{QCoh}(G/H)^G$. If we take $(H \to k) \in BH(k)$, then the corresponding element of $[G\backslash(G/H)](k)$ is $(G \to k, G \to G/H)$. We have 2-Cartesian square

$$\begin{array}{ccc} G & \longrightarrow & G/H \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & [G\backslash(G/H)] \end{array}$$

Then $V = \mathcal{F} \otimes k(1H)$ where $1H \in G/H(k)$. To find the action of H on V, take $H \times H \overset{m,p_2}{\to} H \times H$ in $BH(H)$. This corresponds to $G \times H \overset{m,p_2}{\to} G \times H$ in $[G\backslash(G/H)](H)$. The H-representation on V is gotten from pullback of $\mathrm{act}^*\mathcal{F} \simeq p_2^*\mathcal{F}$ via

$$H \to G \times G/H \rightrightarrows G/H$$

where $H \to G \times G/H$ is inclusion $H \hookrightarrow G$ and $H \to \cdot \to G/H$ corresponding to $1H \in G/H(k)$.

Now start with H-representation V. To get our G-equivariant sheaf on G/H, we need to find out what

$$G/H \to [G\backslash(G/H)] \simeq BH \to \mathrm{QCoh}$$

corresponds to. The atlas $(G \times G/H \overset{\mathrm{act}}{\to} G/H) \in [G\backslash(G/H)](G/H)$ corresponds to left H-bundle $G \to G/H$. Thus our $\mathcal{F} \in \mathrm{QCoh}(G/H)^G$ is given locally by $\mathcal{O}_{S'} \otimes V$ for $S' \to G/H$ a cover trivializing G; the transition maps correspond to action of $H(S' \times_{G/H} S')$ on V as an H-representation. Even though V may be infinite dimensional, we can suggestively write this as

$$\mathcal{F} = G \overset{H}{\times} V.$$

The isomorphism $\mathrm{act}^*\mathcal{F} \simeq p_2^*\mathcal{F}$ should intuitively be given by $G \times G \times^H V \to G \times G \times^H V : (g_1, g_2, v) \mapsto (g_1, g_1 g_2, v)$; this can be made formal using descent and unwinding all the previous constructions (omitted). □

Corollary 4. *Any G-equivariant quasi-coherent sheaf on G/H is flat.*

Proof. From the proof of Corollary 3, we see that on a faithfully flat covering $S' \to G/H$, the pullback of $\mathcal{F}$ is free, hence flat. By faithful flatness we deduce $\mathcal{F}$ is flat. □

2. G-equivariant vector bundles and representations

Let X be a k-scheme with G-action, and suppose $\mathcal{E}$ is a vector bundle with G-equivariant structure on X. The following constructions are taken from [MFK94, Ch. 1, §3].

2.1. G-action on total space $\mathrm{Tot}(\mathcal{E})$.

The the isomorphism $\phi : p_2^*\mathcal{E} \simeq \mathrm{act}^*\mathcal{E}$ gives an isomorphism

$$\phi : (G \times X) \underset{X, p_2}{\times} \mathrm{Tot}(\mathcal{E}) \simeq (G \times X) \underset{X, \mathrm{act}}{\times} \mathrm{Tot}(\mathcal{E})$$

over $G \times X$. The latter is equivalent to the Cartesian square

$$\begin{array}{ccc} G \times \mathrm{Tot}(\mathcal{E}) & \xrightarrow{p_2 \circ \phi} & \mathrm{Tot}(\mathcal{E}) \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \pi} & & \Big\downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\mathrm{act}} & X \end{array}$$

One can check that the above $G \times \mathrm{Tot}(\mathcal{E}) \to \mathrm{Tot}(\mathcal{E})$ is a G-action.

2.2. Making $\Gamma(X, \mathcal{E})$ an $\mathcal{O}_G$-comodule.

We have the following map

$$\Delta : \Gamma(X, \mathcal{E}) \overset{\mathrm{act}^*}{\to} \Gamma(G \times X, \mathrm{act}^*\mathcal{E}) \overset{\Gamma(\phi)}{\to} \Gamma(G \times X, p_2^*\mathcal{E}) \simeq \mathcal{O}_G \underset{k}{\otimes} \Gamma(X, \mathcal{E})$$

where the last step uses projection formula: $p_{2*}p_2^*\mathcal{E} \simeq \mathcal{E} \otimes_{\mathcal{O}_X} p_{2*}\mathcal{O}_{G \times X} \simeq \mathcal{E} \otimes_k \mathcal{O}_G$. The co-cycle condition on ϕ ensures this makes $\Gamma(X, \mathcal{E})$ into a comodule.

2.2.1. *Explicit description.* As described in [Jan03, I.2.8], an $\mathcal{O}_G$-comodule structure on V is equivalent to a group action $G \times \mathbf{V} \to \mathbf{V}$ of schemes, from the sheaf perspective. Let $V := \Gamma(X, \mathcal{E})$. We describe the action $G(k) \times \mathbf{V}(k) \to \mathbf{V}(k)$, where $\mathbf{V}(k) = V$. Given $g \in G(k) \simeq \mathrm{Hom}_k(\mathcal{O}_G, k)$, the action of g on V is just [1]

$$V \overset{\Delta}{\to} \mathcal{O}_G \underset{k}{\otimes} V \overset{g^{-1} \otimes \mathrm{id}}{\to} V.$$

An element of $V = \Gamma(X, \mathcal{E})$ corresponds to a section $f : X \to \mathrm{Tot}(\mathcal{E})$. Then the image in $\Gamma(G \times X, \mathrm{act}^*\mathcal{E})$ is the section $\mathrm{id} \times f : G \times X \to (G \times X) \times_{X, \mathrm{act}} \mathrm{Tot}(\mathcal{E})$, so we have

$$\begin{array}{ccc} (G \times X) \underset{X, \mathrm{act}}{\times} \mathrm{Tot}(\mathcal{E}) & \longrightarrow & \mathrm{Tot}(\mathcal{E}) \\ {\scriptstyle \mathrm{id} \times f}\Big\uparrow & & \Big\uparrow{\scriptstyle f} \\ G \times X & \xrightarrow{\mathrm{act}} & X \end{array}$$

[1] [MFK94] defines a "left"-comodule, while [Jan03] defines a "right"-comodule. Our $\mathcal{O}_G$-comodule structure on $\Gamma(X, \mathcal{E})$ follows [MFK94], so is a "left"-comodule. Thus we need to take g^{-1} when looking at $G \times \mathbf{V} \to \mathbf{V}$.

Applying ϕ gives us a section $f' : G \times X \to G \times \mathrm{Tot}(\mathcal{E})$. Then the image of f' under $\Gamma(G \times X, p_2^*\mathcal{E}) \stackrel{g^{-1} \otimes \mathrm{id}}{\to} \Gamma(X, \mathcal{E})$ is the section making

$$\begin{array}{ccc} \mathrm{Tot}(\mathcal{E}) & \xrightarrow{g^{-1} \times \mathrm{id}} & G \times \mathrm{Tot}(\mathcal{E}) \\ {\scriptstyle g.f}\uparrow & & \uparrow{\scriptstyle f'} \\ X & \xrightarrow{g^{-1} \times \mathrm{id}} & G \times X \end{array}$$

commute. We therefore have commutative diagram

$$\begin{array}{ccccc} \mathrm{Tot}(\mathcal{E}) & \xrightarrow{g} & \mathrm{Tot}(\mathcal{E}) & \xrightarrow{g^{-1}} & \mathrm{Tot}(\mathcal{E}) \\ {\scriptstyle g.g^{-1}.f=f}\uparrow & & {\scriptstyle g.f}\uparrow & & \uparrow{\scriptstyle f} \\ X & \xrightarrow{g} & X & \xrightarrow{g^{-1}} & X \end{array}$$

Thus $g.f = g \circ f \circ g^{-1}$, where we appeal to G-actions on $\mathrm{Tot}(\mathcal{E})$ and X.

2.3. Relating $\Gamma(X, \mathcal{E})$ and fibers of $\mathcal{E}$.

Lemma 5. *Let $x \in X(k), g \in G(k)$. We have commutative square*

$$\begin{array}{ccc} \Gamma(X, \mathcal{E}) & \xrightarrow[\sim]{(g \otimes \mathrm{id})\Delta} & \Gamma(X, \mathcal{E}) \\ \downarrow & & \downarrow \\ \mathcal{E} \otimes k_{g.x} & \xrightarrow[\sim]{(g,x)^*\phi} & \mathcal{E} \otimes k_x \end{array}$$

where g corresponds to ring map $\mathcal{O}_G \to k$.

Proof. To start with, we have $\phi : \mathrm{act}^*\mathcal{E} \simeq p_2^*\mathcal{E}$ on $G \times X$. This gives commutative square

$$\begin{array}{ccc} \Gamma(G \times X, \mathrm{act}^*\mathcal{E}) & \xrightarrow{\Gamma(G \times X, \phi)} & \Gamma(G \times X, p_2^*\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathcal{E} \otimes k_{g.x} = (g,x)^*\mathrm{act}^*\mathcal{E} & \xrightarrow{(g,x)^*\phi} & (g,x)^*p_2^*\mathcal{E} = \mathcal{E} \otimes k_x \end{array}$$

The right map $\Gamma(G \times X, p_2^*\mathcal{E}) \simeq \mathcal{O}_G \otimes \Gamma(X, \mathcal{E}) \to \mathcal{E} \otimes k_x$ factors through

$$\mathcal{O}_G \otimes \Gamma(X, \mathcal{E}) \stackrel{g \otimes \mathrm{id}}{\to} \Gamma(X, \mathcal{E}).$$

Composing the left arrow by $\Gamma(X, \mathcal{E}) \to \Gamma(G \times X, \mathrm{act}^*\mathcal{E})$ gives the claim, since $(g \otimes \mathrm{id})\Delta$ is an isomorphism by properties of comodules [check]. □

3. Line bundles on G/B

For the remainder of these notes, assume k is algebraically closed, and G is a connected reductive linear algebraic group over k. Fix a maximal torus T and Borel subgroup $T \subset B \subset G$.

3.1. Take a weight $\lambda \in X^*(T)$ and consider the corresponding $B \to \mathbf{G}_m$, which is a 1-dimension B-representation; we denote it k_λ. By Corollary 3, the associated bundle

$$\mathcal{O}(-\lambda) := G \overset{B}{\times} k_\lambda$$

is a line bundle on G/B with G-equivariant structure. For open $U \subset G/B$, we have the preimage of U under $G \times^B k_\lambda \to G/B$ is $\pi^{-1}U \times^B k_\lambda$ where is projection $\pi : G \to G/B$. Now $\pi^{-1}U \times^B k_\lambda \simeq [(\pi^{-1}U \times k_\lambda)/B]$ as the stack quotient (B acts on $\pi^{-1}U \times k_\lambda$ by $(g,x).b = (gb, b^{-1}x)$). Thus a section $U \to \pi^{-1}U \times^B k_\lambda$ is the same as a B-bundle $\mathcal{P}$ on U with B-equivariant map

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \pi^{-1}U \times k_\lambda \\ \downarrow & \swarrow_{p_1} & \\ U & & \end{array}$$

This means $\mathcal{P} \to \pi^{-1}U$ is a B-equivariant map of B-bundles, hence an isomorphism. Thus we can normalize to assume $\mathcal{P} = \pi^{-1}U$. Thus a section is the same as a map $f : \pi^{-1}U \to k_\lambda$ where $f(gb) = \lambda(b)^{-1}f(g)$ for $g \in (\pi^{-1}U)(k), b \in B(k)$. It is enough to specify this on k-points since everything is reduced. In general, we would use scheme points. Note that f is just an element of $\Gamma(\pi^{-1}U, \mathcal{O})$. We have shown that

$$\Gamma(U, \mathcal{O}(-\lambda)) = \{f \in \Gamma(\pi^{-1}U, \mathcal{O}) \mid f(gb) = \lambda(b)^{-1}f(g) \text{ for } g \in (\pi^{-1}U)(k), b \in B(k)\}$$

See [Jan03, I.5.8] for another discussion of this.

With this explicit description of the sections of $\mathcal{O}(\lambda)$, we easily see that for $\lambda, \mu \in X^*(T)$, there is isomorphism

$$\mathcal{O}(\lambda) \otimes \mathcal{O}(\mu) \simeq \mathcal{O}(\lambda+\mu) : f \in \Gamma(U, \mathcal{O}(\lambda)), g \in \Gamma(U, \mathcal{O}(\mu)) \mapsto f \cdot g \in \Gamma(U, \mathcal{O}(\lambda+\mu)).$$

This is discussed in [Jan03, II.4.1].

Section 2 details how $V^\lambda := \Gamma(G/B, \mathcal{O}(\lambda))$ is given structure of a G-representation. We have the following theorem:

Theorem 6 ([Spr09, Theorem 8.5.8], [Jan03, Proposition II.2.6])**.** *The weight $\lambda \in X^*(T)$ is dominant if and only if $V^\lambda \neq 0$.*

The dominant weights $X^*(T)^+ = \{\lambda \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+\}$ with respect to B.

Lemma 7. *For $\lambda \in X^*(T)^+$, the line bundle $\mathcal{O}(\lambda)$ is generated by global sections.*

Proof. By Theorem 6, we know $V^\lambda \neq 0$, so take a nonzero global section f. The locus of isomorphy of $\mathcal{O}_{G/B} \xrightarrow{f} \mathcal{O}(\lambda)$ is open and must be nonempty since f is nonzero. Thus there exists $gB \in (G/B)(k)$ such that $f|_{gB} \in \mathcal{O}(\lambda) \otimes k_{gB}$ is nonzero. This implies $V^\lambda \to \mathcal{O}(\lambda) \otimes k_{gB}$ is surjective since the latter is 1-dimensional k-vector space. As $G(k)$ acts transitively on $(G/B)(k)$, Lemma 5 implies $V^\lambda \to \mathcal{O}(\lambda) \otimes k_x$ is surjective for all $x \in (G/B)(k)$. We have shown $V^\lambda \otimes_k \mathcal{O}_{G/B} \to \mathcal{O}(\lambda)$ is surjective on closed points, hence surjective. □

Let $\lambda \in X^*(T)^+$. We now know that $V^\lambda \twoheadrightarrow \mathcal{O}(\lambda) \otimes k_{1B}$, so taking duals, we get $\ell^\lambda \subset (V^\lambda)^\vee$ where $\ell^\lambda := (\mathcal{O}(\lambda) \otimes k_{1B})^\vee$ is a B-stable one-dimensional subspace by Lemma 5. As $\mathcal{O}(\lambda) \otimes k_{1B}$ is isomorphic as a B-representation to $k_{-\lambda}$ (cf. proof of Corollary 3), we have B acts on ℓ^λ via λ.

Before proceeding, we expound on a comment of [Jan03, II.1.3, pg. 176].

Lemma 8. *For any $\alpha \in R$, there exists a homomorphism*

$$\varphi_\alpha : \mathrm{SL}_2 \to G$$

such that for a suitable normalization of $u_{\pm\alpha}$:

$$\varphi_\alpha \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = u_\alpha(x), \quad \varphi_\alpha \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = u_{-\alpha}(x), \quad \textit{and} \quad \varphi_\alpha \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = \alpha^\vee(x)$$

for $x \in k$.

For other properties satisfied by φ_α, see the same page of [Jan03].

Proof. Refer to the proof of [Spr09, Lemma 8.1.4]. We have $U_\alpha \subset (G_\alpha, G_\alpha)$, where the latter is semi-simple of rank one. The claim then follows from the proof of [Spr09, Theorem 7.2.4]. The last equality requires the description of $\alpha^\vee$ given in the proof of [Spr09, Theorem 7.3.5]; alternatively we can just use [Spr09, Lemma 8.1.4]. □

Lemma 9. *Suppose we have a finite dimensional representation $\phi : G \to \mathrm{GL}(V)$. For $v \in V$, set $P = \mathrm{Stab}_G(kv)$. We have $i : G/P \hookrightarrow \mathbf{P}(V)$ is a locally closed embedding. Let $\lambda : P \to \mathbf{G}_m$ describe the action of P on kv. Then $i^*\mathcal{O}(1) \simeq \mathcal{O}_{G/P}(\lambda)$ as G-equivariant sheaves.*

Proof. We have $\mathrm{GL}(V)$ acts on $\mathbf{P}(V)$. An element $s \in \mathbf{P}(V)(S)$ is $\mathcal{L} \hookrightarrow V \otimes \mathcal{O}_S$. So for $g \in \mathrm{GL}(V)(S)$, take $g.s$ to be $\mathcal{L} \hookrightarrow V \otimes \mathcal{O}_S \xrightarrow{g} V \otimes \mathcal{O}_S$. Then we have $(g.s)^*\mathcal{O}(-1) \simeq \mathcal{L} \simeq s^*\mathcal{O}(-1)$, which provides $\mathcal{O}(-1)$ a $\mathrm{GL}(V)$-equivariant structure. Consequently, all $\mathcal{O}(n), n \in \mathbf{Z}$ are $\mathrm{GL}(V)$-equivariant.

We check what the P-representation on the fiber of $i^*\mathcal{O}(-1)$ at $1P \in (G/P)(k)$ is. Taking $g \in P(S)$, we have $1P \in \mathbf{P}(V)(k)$ is $kv \hookrightarrow V$, and $g.1P$ is

$$kv \otimes \mathcal{O}_S \hookrightarrow V \otimes \mathcal{O}_S \xrightarrow{\phi(g)} V \otimes \mathcal{O}_S.$$

The action on $kv \otimes \mathcal{O}_S$ is λ by definition:

$$\begin{array}{ccc} kv \otimes \mathcal{O}_S & \hookrightarrow & V \otimes \mathcal{O}_S \\ \downarrow{\scriptstyle \lambda(g)} & & \downarrow{\scriptstyle \phi(g)} \\ kv \otimes \mathcal{O}_S & \hookrightarrow & V \otimes \mathcal{O}_S \end{array}$$

Thus $i^*\mathcal{O}(-1)$ corresponds to λ under the equivalence of Corollary 3. Taking the dual, we have $i^*\mathcal{O}(1)$ corresponds to $-\lambda$, i.e., $i^*\mathcal{O}(1) \simeq \mathcal{O}_{G/P}(\lambda)$. □

In particular, if in Lemma 9 we take the standard representation on $G = \mathrm{GL}(V)$, then $G/P \to \mathbf{P}(V)$ is surjective on k-points, hence an isomorphism. Letting ω be the action of P on kv, we have $\mathcal{O}(1) \simeq \mathcal{O}_{G/P}(\omega)$ as in [Jan03, II.4.3(1)].

Remark 10. If G is a non-reduced affine algebraic group over k, then since k is algebraically closed, hence perfect, G_{red} is a reduced algebraic subgroup of G [Mil10, Proposition I.5.11]. Now G_{red} is smooth over k, while G is not. By the definition of smoothness via regularity, G is smooth iff the tangent spaces T_xG all have dimension $\dim G$ for $x \in G(k)$. By homogeneity, $\dim T_xG = \dim \mathrm{Lie}(G)$. We deduce that $\mathrm{Lie}(G_{\mathrm{red}}) \subsetneq \mathrm{Lie}(G)$ is never an equality.

Lemma 11. *Let $\lambda \in X^*(T)$. Then $\mathcal{O}(\lambda)$ is very ample iff $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \in R^+$.*

Proof. Proof taken from [Jan03, Proposition 4.4]. Suppose $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \in R^+$. By Chevalley's theorem, we know there is some G-module V and $v \in V$ such that $\mathrm{Stab}_G(kv) = B$. Let $\mu \in X^*(T)$ be the character corresponding to B-action on kv. Then there is positive integer m such that $m\lambda - \mu \in X^*(T)^+$. Let $V' = (V^{m\lambda-\mu})^\vee$, which is a nonzero G-module by Theorem 6. We noted earlier that there is $v' \in V'$ such that $B \subset \mathrm{Stab}_G(kv')$ and B acts on kv' by $m\lambda - \mu$. Now $V \otimes V'$ is a G-module and B acts on $k(v \otimes v')$ by $m\lambda$. By choosing a basis, one sees that $B(k) = \mathrm{Stab}_G(k(v \otimes v'))(k)$ and $\mathrm{Lie}(B) = \mathrm{Lie}(\mathrm{Stab}_G(k(v \otimes v')))$. Remark 10 implies

$$B = \mathrm{Stab}_G(k(v \otimes v'))$$

Lemma 9 says the inverse image of $\mathcal{O}(1)$ under $G/B \hookrightarrow \mathbf{P}(V \otimes V')$ is isomorphic to $\mathcal{O}_{G/B}(m\lambda)$. Hence $\mathcal{O}(m\lambda) \simeq \mathcal{O}(\lambda)^{\otimes m}$ is very ample, so $\mathcal{O}(\lambda)$ is ample.

For the proof of very ampleness, see [Jan03, II.8.5(1)].

In the other direction, assume $\mathcal{O}(\lambda)$ is ample. For $\alpha \in R^+$, we have $\varphi_\alpha : \mathrm{SL}_2 \to G$ by Lemma 8. Let $B' \subset \mathrm{SL}_2$ be the Borel subgroup of upper triangular matrices. Looking at $B'(k)$ and $\mathrm{Lie}(B')$, we deduce that B' is the stabilizer of $1B \in (G/B)(k)$. By [DG70, III, §3, Proposition 5.2], we have $\bar{\varphi}_\alpha : \mathrm{SL}_2/B' \hookrightarrow G/B$ is a closed embedding. Direct computation shows $\mathrm{SL}_2/B' \simeq \mathbf{P}^1$. Since φ_α on the diagonal equals $\alpha^\vee$, we deduce from Corollary 3 that

$$\bar{\varphi}_\alpha^* \mathcal{O}_{G/B}(\lambda) \simeq \mathcal{O}_{\mathrm{SL}_2/B'}(\langle \lambda, \alpha^\vee \rangle) \simeq \mathcal{O}_{\mathbf{P}^1}(\langle \lambda, \alpha^\vee \rangle),$$

where the latter isomorphism is an application of Lemma 9. Ampleness of $\mathcal{O}_{G/B}(\lambda)$ implies $\mathcal{O}_{\mathbf{P}^1}(\langle \lambda, \alpha^\vee \rangle)$ is ample. Since we know what cohomology on $\mathbf{P}^1$ looks like, we deduce $\langle \lambda, \alpha^\vee \rangle > 0$. □

Proof in characteristic zero. There is a nicer proof of the lemma if k is of characteristic zero because we know stabilizer subgroups are reduced. We can consider the G-module $(V^\lambda)^\vee$ with the one-dimensional subspace $\ell^\lambda := (\mathcal{O}(\lambda) \otimes k(1B))^\vee$. We know B stabilizes ℓ^λ and acts by λ. We claim that $\mathrm{Stab}_G(\ell^\lambda) = B$. For any $\alpha \in R^+$, consider $\varphi_\alpha : \mathrm{SL}_2 \to G$. Then considering $(V^\lambda)^\vee$ as a SL_2-module, we see that ℓ^λ has weight $\langle \lambda, \alpha^\vee \rangle > 0$. By classification of reduced parabolic subgroups of reductive groups [Spr09, Theorem 8.4.3], $\mathrm{Stab}_{\mathrm{SL}_2}(\ell^\lambda)$ is either B' or SL_2. The representation is non-trivial, which rules out the latter case by [Spr09, 7.2.5(2)]. Since this holds for every $\alpha \in R^+$, we know that the only U_α in $\mathrm{Stab}_G(\ell^\lambda)$ are $\alpha \in R^+$. By [Spr09, Theorem 8.4.3] again, we conclude that $\mathrm{Stab}_G(\ell^\lambda) = B$. Now $\mathcal{O}(\lambda)$ is just the pullback of $\mathcal{O}(1)$ under $G/B \hookrightarrow \mathbf{P}((V^\lambda)^\vee)$. □

The above proof does not work in nonzero characteristic: suppose $\mathrm{char}\, k = 2$. Take the SL_2-module V_2 from [Spr09, 7.3.7(1)]. If we let $P = \mathrm{Stab}_{\mathrm{SL}_2}(ke_0)$, then $P(k) = B(k)$ but $\mathrm{Lie}(P) = \mathfrak{sl}_2$. [2]

4. Plücker relations on G/B

Recall the assumptions on G of the previous section. For any $\lambda, \mu \in X^*(T)$, we have $\mathcal{O}(\lambda) \otimes \mathcal{O}(\mu) \simeq \mathcal{O}(\lambda + \mu)$. Taking global sections gives a canonical map $V^\lambda \otimes V^\mu \to V^{\lambda+\mu}$. We prove the following result.

[2] If $\mathrm{char}\, k = p \neq 0$, then $P_n := \mathrm{Spec}\, k[x, y, z, w]/(z^{p^n}, xw - yz - 1)$ is a subgroup of SL_2 containing B' and is non-reduced for $n > 0$.

Theorem 12. *As a functor, G/B may be described as*

$$\mathrm{Hom}(S, G/B) = \{\mathcal{L}^\lambda \in \mathrm{Pic}(S) \text{ for all } \lambda \in X^*(T)^+,\ V^\lambda \otimes \mathcal{O}_S \twoheadrightarrow \mathcal{L}^\lambda \text{ satisfying Plücker relations}\}$$

where the Plücker relations require there to be, for $\lambda, \mu \in X^*(T)^+$, an isomorphism $\mathcal{L}^\lambda \otimes \mathcal{L}^\mu \simeq \mathcal{L}^{\lambda+\mu}$ such that

$$\begin{array}{ccc} (V^\lambda \otimes \mathcal{O}_S) \otimes_{\mathcal{O}_S} (V^\mu \otimes \mathcal{O}_S) & \twoheadrightarrow & \mathcal{L}^\lambda \otimes \mathcal{L}^\mu \\ \downarrow & & \downarrow \wr \\ V^{\lambda+\mu} \otimes \mathcal{O}_S & \twoheadrightarrow & \mathcal{L}^{\lambda+\mu} \end{array}$$

commutes.

Theorem 12 generalizes to the following.

Theorem 13. *Let $\mathcal{P}$ be a G-bundle on X. Then*

$$\mathrm{Hom}_X(S, \mathcal{P}/B) = \{\mathcal{L}^\lambda \in \mathrm{Pic}(S) \text{ for all } \lambda \in X^*(T)^+,\ (V^\lambda)_{\mathcal{P}} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_S \twoheadrightarrow \mathcal{L}^\lambda \text{ satisfying Plücker relations}\}.$$

4.1. Take a dominant weight λ large enough so that $\mathcal{O}(\lambda)$ is very ample. By Lemma 7, we have $V^\lambda \otimes \mathcal{O}_{G/B} \to \mathcal{O}(\lambda)$ is surjective. Let the kernel be $\mathcal{F}$. Then ampleness implies $H^1(G/B, \mathcal{F} \otimes \mathcal{O}(m\lambda)) = 0$ for all $m \gg 0$. Therefore by the long exact sequence

$$V^\lambda \otimes V^{m\lambda} \simeq \Gamma(G/B, V^\lambda \otimes \mathcal{O}(m\lambda)) \to \Gamma(G/B, \mathcal{O}((m+1)\lambda)) \simeq V^{(m+1)\lambda}$$

is surjective for $m \gg 0$. Thus if we replace λ by some $m\lambda$, we can assume that

$$\mathrm{Sym}_k(V^\lambda) \to \bigoplus_{n \geq 0} V^{n\lambda}$$

is surjective. Note that $V^0 = k$ and $\bigoplus V^{n\lambda}$ is a $\mathbf{Z}_{\geq 0}$ graded k-algebra.

Lemma 14. *Let X be a projective scheme over k with very ample line bundle $\mathcal{L}$. Then*

$$X \simeq \mathrm{Proj} \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n).$$

Proof. Take some closed embedding $i : X \to \mathbf{P}(V)$ with $i^*\mathcal{O}(1) \simeq \mathcal{L}$. Let $A := \mathrm{Sym}_k(V^\vee)$. The ideal sheaf of i is of form $\widetilde{\mathrm{Loc}}(I)$ for a homogeneous ideal $I \subset A$. We have $X \simeq \mathrm{Proj}(A/I)$. By projection formula we have $i_*i^*\mathcal{O}(n) \simeq \mathcal{O}(n) \otimes i_*i^*\mathcal{O}$. Note that $i_*i^*\mathcal{O} \simeq \widetilde{\mathrm{Loc}}(A/I)$. Since $\Gamma(X, \mathcal{L}^n) \simeq \Gamma(\mathbf{P}(V), i_*(\mathcal{L}^n))$, we have

$$\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n) \simeq \widetilde{\Gamma}(i_*i^*\mathcal{O}) \simeq \widetilde{\Gamma}\,\widetilde{\mathrm{Loc}}(A/I)$$

By a corollary of Serre's theorem, we have A/I and $\widetilde{\Gamma}\,\widetilde{\mathrm{Loc}}(A/I)$ agree in degrees $\gg 0$. Thus they have the same Proj, which proves the claim. □

Applying Lemma 14 to G/B and $\mathcal{O}(\lambda)$, we get

(1) $$G/B \simeq \mathrm{Proj} \bigoplus V^{n\lambda}.$$

We have the following description of mapping schemes into $\mathrm{Proj}\, A$.

Lemma 15 (233A.10.7)**.** *Let A be a non-negatively graded A_0-algebra, such that $\mathrm{Sym}_{A_0}(A_1) \to A$ is surjective in degrees $\gg 0$. Assume A_0 is Noetherian and A_1 is finitely generated as an A_0-module.*

For any scheme S, $\mathrm{Hom}(S, \mathrm{Proj}\, A)$ consists of triples $(\Phi, \mathcal{L}, \alpha)$ where $\Phi : S \to \mathrm{Spec}\, A_0$, $\mathcal{L}$ is a line bundle on S, and α is a homomorphism of quasi-coherent sheaves of graded algebras on S

$$\alpha : \Phi^*(\mathcal{A}) \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$$

which is surjective in degrees $\gg 0$. We identify two triples if the lines bundles are isomorphic and the α's agree in high degrees (in a compatible way). Here $\mathcal{A} = \mathrm{Loc}\, A$ is the quasi-coherent sheaf on $\mathrm{Spec}\, A_0$.

Details and further generalities of this lemma can be found in [Gro61, II.3.7] or [sta].

Proof. First note that it will suffice to prove the claim when S is affine (use a gluing argument). Under the assumptions, $\mathcal{O}_{\mathrm{Proj}\, A}(1)$ is a very ample line bundle (see 233A.10.6). Suppose we have a map $f : S \to \mathrm{Proj}\, A$. Since A_1 is finitely generated over A_0, we have

$$A \to \bigoplus_{n \geq 0} \Gamma(\mathrm{Proj}\, A, \mathcal{O}(n))$$

is an isomorphism in high degree (see 233A.10.6). By ampleness, $\mathcal{O}(n)$ is generated by global sections for $n \gg 0$. Thus if we let π denote $\mathrm{Proj}\, A \to \mathrm{Spec}\, A_0$, we have

$$\alpha : \pi^* \mathcal{A} \to \bigoplus_{n \geq 0} \pi^* \pi_* \mathcal{O}(n) \to \bigoplus_{n \geq 0} \mathcal{O}(n)$$

is surjective in high degree. Applying f^* and setting $\mathcal{L} = f^* \mathcal{O}_{\mathrm{Proj}\, A}(1)$ gives $(\Phi = \pi \circ f, \mathcal{L}, \alpha)$.

In the other direction, suppose we are given $(\Phi, \mathcal{L}, \alpha)$. Recall that we are assuming $S = \mathrm{Spec}\, B$ is affine. Thus α is a map of graded B-algebras

$$B \underset{A_0}{\otimes} A \to \bigoplus_{n \geq 0} \mathcal{L}^n =: A'$$

We can compose with $A \to B \otimes_{A_0} A$ to get map $A \to A'$ of graded A_0-algebras. Recall that a map of graded algebras $A \to A'$ induces a map of Proj iff $A'^+ \subset \mathrm{rad}(A' A^+)$. This is satisfied since $A'_0 = B$ and we assume α is surjective in high degrees. Thus we have a map

$$\mathrm{Proj}\, A' \to \mathrm{Proj}\, A.$$

We claim the canonical projection $\mathrm{Proj}\, A' \to \mathrm{Spec}\, B$ is an isomorphism. This can be checked locally, so we reduce to $\mathcal{L} \simeq B$ free. Then $\mathrm{Proj}\, A' \simeq \mathrm{Proj}\, B[T] \simeq \mathbf{P}^0_B \simeq \mathrm{Spec}\, B$. We have thus constructed a map

$$f : \mathrm{Spec}\, B \simeq \mathrm{Proj}\, A' \to \mathrm{Proj}\, A$$

The pullback of $\mathcal{O}_{\mathrm{Proj}\, A}(1)$ along $\mathrm{Proj}\, A' \to \mathrm{Proj}\, A$ is $\mathcal{O}_{\mathrm{Proj}\, A'}(1)$, which can be checked to coincide with $\mathcal{L}$ on $\mathrm{Spec}\, B$ (check on open affine subsets of $\mathrm{Spec}\, B$ with $\mathcal{L}$ is trivialized, and note that the transition maps for Čech cocycles coincide).

Lastly suppose we start with $f : \operatorname{Spec} B \to \operatorname{Proj} A$ and we want to check that

$$\begin{array}{ccc} \operatorname{Proj} A' & & \\ \downarrow{\scriptstyle \sim} & \overset{\tilde{f}}{\searrow} & \\ \operatorname{Spec} B & \xrightarrow{f} & \operatorname{Proj} A \end{array}$$

commutes. This is a local assertion, so take $a \in A_1$ and consider $\operatorname{Spec}(A_a)_0 \subset \operatorname{Proj} A$ on which $\mathcal{O}_{\operatorname{Proj} A}(1)$ is trivialized. Take $\operatorname{Spec} B_b \subset f^{-1}(\operatorname{Spec}(A_a)_0)$. By functoriality, it now suffices to check

$$\begin{array}{cccc} \operatorname{Proj} \bigoplus (\mathcal{L}_b)^n & & & \\ \downarrow{\scriptstyle \sim} & \searrow & & \\ \operatorname{Spec} B_b & \xrightarrow{f|} & \operatorname{Spec}(A_a)_0 & \hookrightarrow \operatorname{Proj} A \end{array}$$

commutes. Now $\mathcal{L}_b \simeq B_b$ is free. With a little work, this can be checked directly from constructions.

This shows our above two maps are mutually inverse, which proves the claim. □

We can finally prove Theorem 12.

Proof of Theorem 12. Given a map $f : S \to G/B$, we set $\mathcal{L}^\lambda := f^*\mathcal{O}(\lambda)$. By Lemma 7, we have $V^\lambda \otimes \mathcal{O}_{G/B} \twoheadrightarrow \mathcal{O}(\lambda)$ is surjective, so applying f^* gives $V^\lambda \otimes \mathcal{O}_S \twoheadrightarrow \mathcal{L}^\lambda$. The Plücker relations are satisfied because they are on G/B.

Conversely if we are given $V^\lambda \otimes \mathcal{O}_S \twoheadrightarrow \mathcal{L}^\lambda$ satisfying Plücker relations, if we take a particular λ such that (1) holds, then Lemma 15 gives a map $S \to G/B$. (Note that we don't actually need the data of $V^\lambda \otimes \mathcal{O}_S \twoheadrightarrow \mathcal{L}^\lambda$ for all $\lambda \in X^*(T)^+$, only the multiples of our chosen λ.)

The proof of Lemma 15 shows that the two maps just described are inverse (and that the latter map is independent of choice of λ). □

Proof of Theorem 13. This follows from Theorem 12 by descent. Add details. □

5. G/N is quasi-affine

To complete the picture of the flag variety G/B, we show that G/N is quasi-affine, where $N = B_u$ is the unipotent part of our Borel.

5.1. Universal property of V^λ.

As an aside, we talk a little about the representation theory of G and B. This can all be found in [Jan03, Ch. 3]

Definition 16. Let V be a B-representation. Then $\operatorname{coind}_B^G(V)$, if it exists, is the G-representation that is left adjoint to the forgetful functor, i.e., there exists natural isomorphism

$$\operatorname{Hom}_B(V, W) \simeq \operatorname{Hom}_G(\operatorname{coind}_B^G(V), W)$$

for all G-representations W.

Analogously, $\operatorname{ind}_B^G(V)$ is the G-representation that is right adjoint to the forgetful functor, i.e.,

$$\operatorname{Hom}_B(W, V) \simeq \operatorname{Hom}_G(W, \operatorname{ind}_B^G(V))$$

for all G-representations W.

Lemma 17. *Let $\lambda \in X^*(T)$. Then $\operatorname{ind}_B^G(k_{-\lambda}) \simeq V^\lambda$.*

Proof. Recall from Section 3 that

$$V^\lambda = \{f \in \mathcal{O}_G \mid f(gb) = \lambda(b)f(g) \text{ for } g \in G(k), b \in B(k)\}.$$

From the explicit description of the G-representation on V^λ given in Section 2.2.1, we have $g.f(x) = f(g^{-1}x)$, i.e., G acts on $V^\lambda \subset \mathcal{O}_G$ by left translation. Define

$$\mathrm{Hom}_G(W, V^\lambda) \to \mathrm{Hom}_B(W, k_{-\lambda})$$

by taking ϕ to the map $w \mapsto \phi(w)(1)$. Conversely, given $\xi \in \mathrm{Hom}_B(W, k_{-\lambda})$, set $\phi : W \to V^\lambda \subset \mathcal{O}_G$ to be $\phi(w)(g) = \xi(g^{-1}.w)$. The two given maps are inverses, so we have the desired natural isomorphism. □

Lemma 18. *Let* $\lambda \in X^*(T)$*. Then* $\mathrm{coind}_B^G(k_\lambda) \simeq \mathrm{ind}_B^G(k_{-\lambda})^\vee$.

Proof. First take W to be a finite dimensional G-representation. Then we have

$$\begin{aligned}\mathrm{Hom}_G(\mathrm{ind}_B^G(k_{-\lambda})^\vee, W) &\simeq \mathrm{Hom}_G(W^\vee, \mathrm{ind}_B^G(k_{-\lambda})) \\ &\simeq \mathrm{Hom}_B(W^\vee, k_{-\lambda}) \simeq \mathrm{Hom}_B(k_\lambda, W).\end{aligned}$$

These isomorphisms are natural. Now since any G-representation W is a direct limit of finite dimensional G-representations [MFK94, §1, Lemma], and $\mathrm{ind}_B^G(k_{-\lambda}) \simeq V^\lambda$ (Lemma 17) is finite dimensional, we deduce that the above isomorphism holds for any G-representation W. □

We have the canonical map $V^\lambda \to k_{-\lambda} : f \mapsto f(1)$. The isomorphism in the above proof is just given by precomposition by the dual map $k_\lambda \to (V^\lambda)^\vee$.

5.2.

Lemma 19. *The ring of global sections admits a grading*

$$\Gamma(G/N, \mathcal{O}_{G/N}) \simeq \bigoplus_{\lambda \in X^*(T)^+} V^\lambda$$

where the ring multiplication $V^\lambda \otimes V^\mu \to V^{\lambda+\mu}$ *coincides with the map in the Plücker relations.*

Here we are considering rings graded in a commutative semigroup.

Proof. Since G/N is a geometric quotient, we have $\Gamma(G/N, \mathcal{O}_{G/N}) = (\mathcal{O}_G)^N$ where we consider the regular representation on $\mathcal{O}_G$ by right translations. Since N is normal in B, the regular representation of T on $\mathcal{O}_G$ extends to a T-representation on $(\mathcal{O}_G)^N$ [Jan03, Lemma 3.2]. We can therefore decompose $(\mathcal{O}_G)^N$ into weight spaces. For $\lambda \in X^*(T)$, we get

$$\Gamma(G/N, \mathcal{O})_\lambda = ((\mathcal{O}_G)^N)_\lambda \simeq \mathrm{Hom}_B(k_\lambda, \mathcal{O}_G) \simeq \mathrm{Hom}_G(\mathrm{coind}_B^G(k_\lambda), \mathcal{O}_G).$$

Composition with $\mathcal{O}_G \to k : f \mapsto f(1)$ gives an isomorphism

$$\mathrm{Hom}_G(\mathrm{coind}_B^G(k_\lambda), \mathcal{O}_G) \simeq \mathrm{coind}_B^G(k_\lambda)^\vee.$$

Given $\xi \in \mathrm{coind}_B^G(k_\lambda)^\vee$, we define $\phi : \mathrm{coind}_B^G(k_\lambda) \to \mathcal{O}_G$ by $\phi(v)(g) = \xi(g.v)$. Now Lemmas 17, 18 imply

$$\Gamma(G/N, \mathcal{O})_\lambda \simeq V^\lambda.$$

One can chase the maps defined in the lemmas above (I have done this) to see that the above isomorphism is the same as the equality $\Gamma(G/N, \mathcal{O})_\lambda = V^\lambda$ considered inside $\mathcal{O}_G$.

From the equality inside $\mathcal{O}_G$, the assertion concerning multiplication is obvious given the discussion of tensor products at the beginning of Section 3. □

Let $A := \Gamma(G/N, \mathcal{O})$. Lemma 19 shows A is a $X^*(T)^+$ graded ring, where $X^*(T)^+$ is a commutative semigroup.

Definition 20. A scheme X is *quasi-affine* if the canonical map $X \to \operatorname{Spec} \Gamma(X, \mathcal{O})$ is a quasi-compact open embedding.

Lemma 21. *The quotient G/N is quasi-affine.*

Proof. From Chevalley's theorem, we know there exists a finite dimensional representation V with $v \in V$ such that $N = \operatorname{Stab}_G(kv)$. Then N must act on kv by some character. Since N is unipotent, this character is trivial. Therefore we have have $N = \operatorname{Stab}_G(v)$, which gives a locally closed embedding $G/N \hookrightarrow \mathbf{V}$. Therefore G/N is an open subscheme of an affine variety. By [sta, Lemma 19.13.4], G/N is quasi-affine. □

Note that the map $G \to \operatorname{Spec} A$ corresponding to $A \subset \mathcal{O}_G$ is dominant and factors through $G \to G/N \to \operatorname{Spec} A$. Hence $G/N \to \operatorname{Spec} A$ is a dominant open embedding.

Since T normalizes N in B, we have $(G/N)/T \simeq G/B$. So there is an open of $\operatorname{Spec} A$ whose quotient under T-action is isomorphic to the flag variety G/B.

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